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Original article

On Frenet Framme of Null Curves in Three Dimensional Minkowski Spaces

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Abstract
This article investigates the Frenet frame of null curves which are embedded inside three-dimensional Minkowski spacetime M^2,1, a significant topic of differential geometry and Lorentzian geometry. Unlike Euclidean geometry, the indefinite metric of Minkowski space introduces unique challenges and properties. We analyse the Frenet frame components, including the tangent of curves, and normal which is orthogonal, and binormal this is a perpendicular to both vectors, and derive the associated Frenet-Serret equations. Additionally, we propose new formulations for these equations specific to null curves in the Minkowski space. A practical example is visualized using Maple to illustrate the theoretical results. This study contributes to the deeper understanding of the geometric behaviour of null curves in Lorentzian settings
Keywords. Multilinear Algebra, Minkowski Spaces, Cross Product of 3d Minkwoski Spaces, Lorentz Transformation.

INTRODUCTION

Recently the language of Minkowski spaces appears in the differential geometry, this space can deal with symmetric operation of bilinear with signature at least one minus. i.e. (-,+,+,+). Actually, the French mathematician Hermann Minkowski (1864-1909) gives his talk on four dimwnsional vector which equivalent ro vector space we mean the usual one. This talk was in 1908. In mathematical conference at that time. Occasionally, the four-dimensional space of Minkowski that later called the spacetime, always assumes at least one vector which is time. Despite of this, we may assume that the vector is in three dimensions instead of four, and also, we spot into account that there is at least one time. This procedure is equipped with signature of symmetric that is bilinear of (-,+,+). As a result, this paper concerns on this type of vectors in three dimensional replaced the usual one which used already in classical differential geometry.

On the other hand, the differential geometry has appeared as an independent subject in mathematics since a long history. One example is studying the movement curves in space E^3 that is studying a positively definite of othonomal basis that are describe the curve are so-called Frenet trihedron. They are triad to understant orthogonal unit vectors over any arbitrary curve. Called tangent (gradient), normal or perpendicular, and binormal this also known as a perpendicular to both normal and tanget vectors. At any point on a curve in E^3, by using Frenet trihedron we can understand and clearly describe the curvature and torsion of the curve at some point point.

In the last three decades, the studying differential geometry on a spacetime such Minkowski space has been studied and discovers widely. One of which is the curves theory of Minkowski spaces. Here we recall the types of spacetime vectors of Minkowski space; time-like, space-like finally light-like. Which also all have been discovered; see for example [1-3]. This work focuses on the light-like (null) curve in $3D$ Mikowski space, assuming that restriction of one axis, and see how Ferent-Serret formula can carries over up to this case. The example will provide with visualization using Maple 17.

In sections 2 and 3 we produce a background material of this manuscript, and they will cover the curves and Frenet frames in three dimensional Euclidian E^3 and Minkowskian M_1^2 spaces respectively. We then will give a brief explanation of Ferent Frames and there beneficial in describing the curvature of curves. Section 4 which covers the null curves, we in this section will try to recover the light-like (null) curves in M_1^2, therefore we have new formula of Frenet-serret equations.

An explicit example of null curves will be given in section 5 and give in direct way how frenet frames can carry over up to our supposition of null curves. Then we will use the Lorentz transformation to ensure that the curve whether can save its properties over transformations or not. Thus, there is a visualization of this curve in both cases before and after transformations. Finally, we end this section by taking a $2D$ projection to see the same curve and its curvature if it transposed into $2D$ time-like space. It unfortunately goes non-relationship between the curvatures. Then, the conclusion of this paper will be provided, assuming a future work included.

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Methods

Curves in E^3

As a background material of the topic of curves theory, we do need to introduce the following concepts, moreover as a conseuquence we assume the reader is familier with the topic of linear algebra and multilinear of vectors topic, we mean the dual product and dot (inner) and outer (cross) product, and alsi the tensors by mean of usual cross product in three dimensional spaces.
For more elementary material can be enclosed in [4]. However, it will be advantageous to introduce some curves theory ideas in E^3.
A map so called unit speed curve is $\gamma: I \to V$ defined on an open set (may unbounded) $I \subset E$ into a vector three-dimensional space $V \subset E^3$, stating as a differentiable graph function, and assume γ, γ' and γ'' are linearly independent as a usual vector. Furthermore, the curve γ is may called (sometimes) a *regular* if $|\gamma'| \neq 0$ *and so-called unit curve* if $|\gamma'| = 1$. Also, the *arc length* functions are $L_{t_0}^{t}(\gamma) = \int_{t_0}^{t} \sqrt{g(\gamma', \gamma')}du$, so that the map $g(\gamma', \gamma')$ is the dot product of vectors. Assume that $\gamma: I \to V$ is curve whose arc length equals to 1. futhermore the curve might be parametrized by its arc length to provide its a length equals to 1 [5,6]. In all-purpose, hereafter assume any curve is regular and arc length parameterized.

Frenet Trihedron (Frames) in 3D Euclidean E^3

Now, it's known that the, γ' is the *tangent on the curve* , and denoted T(s) , also describe the normal on the curve which is the perpendicular on the tangent by given vector by $N(s) = \frac{T'(s)}{|T'(s)|}$; and in E^3 the curvature is given by $|T'(s)| = g(T'(s), T'(s))$ which then known by the function $\kappa: I \to E$ that is may only continuous. i.e., the important condition is continuous function. But, together $T(s), N(s): I \to E$ shall be differentiable as a function that are also unit maps, we mean that are|T(s)|=1, N(s)|=1. we say $g(T(s), T(s)) = 1 = g(N(s), N(s))$. Futhermore $g(T(s), N(s)) = 0$ each $S \epsilon I$. So far-off, we arrive T'(s)= κ (s) N(s) this a differential equation. Nevertheless in E^3, its appropriate to increase a third vector perpendicular to both two called the *Bilinear* B(s). Well-defined as B(s)= T(s) × N(s), somewhere × is the vector product in Euclidean space E^3. Obviously,$B(s): I \to E$ also differentiable as a function, contents |B(s)|=1, or $g(T(s), T(s)) = 1$ we mean that the Binormal is also has unit of length. and both $g(T(s), B(s)) = 0 = g(N(s), B(s))$; Therefore, now we are reviewing an orthonormal basis {T(s), N(s), B(s)} which are later called the *Frenet trihedron* of arbitrary curve γ.[7]
The *Torsion* $\tau(s): I \to E$ is also defined as a continuous function. then going ahead, we grew B'(s)= - τ(s) N(s). as a conclusion N'(s) can be defined as a combination is both direction of $T(s) and\ B(S)$, given by $N'(s) = \propto (s)T(s) + \beta\, B(s)$ (See [3] P 14-15). Producing that $\propto = -\kappa$ and $= -\tau$. as a result; $N'(s) = \kappa(s)T(s) + \tau\, B(s)$.
To some up, the *Frenet Formula* in Euclidean space E^3 explined by:
T'(s)= κ (s) N(s)
$N'(s) = \kappa(s)T(s) + \tau\,(s)B(s)$
B'(s)= - τ(s) N(s).
Which is in matrix representation by:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa\,(s) & 0 \\ -\kappa\,(s) & 0 & \tau(s) \\ 0 & -\,\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ N(s) \end{pmatrix}$$

Up to now, the main goal of this article is sorting out the Frenet formula inside the Minkowski space $M^{2,1}$, assuming there is a time vector at least one, and studying only the case that is null vector of curvature κ and torsion τ . it just trying to derive the Frenet formula in null vectors by new idea. As we will see below.

Results

Deriving Null-Frenet formula in 3D Minkowski Space $M^{2,1}$

Assuming that the reader has studied as mension before the Minkowski Spacetime, and has a good knowledge of other cases of time moving along Trihedron, that allow him to understand the procedure and what is going in here, we assume the reader has studied at least one or two from those references; Lopez [1], Walter book [2], Ilarslan et al [3] Almoneef et al [12], and Yılmaz et al [13]. But now we only go inside our supposition which is a technique to understand a null moving of curve by mean of Frenet.
Suppose thaty: $E^1 \to M^{2,1}$, such that $\gamma' \neq 0$ but$g(\gamma', \gamma') = 0$; then $T = \gamma'$ is null. And it cannot be normalized by are-length parameterization!
Our idea is fixing the unit time-like vector e_t, and then we can choose the parameter such that:

$$g(e_t, T) = 1$$

Using this parameter, and take

$$\kappa N = T'$$

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and define κ as curvature $\sqrt{g(T',T')}$.
From here, we want to do as much as Frenet Frames [8,9,10].
From our supposition, we know, $g(T,T) = 0$ is light-like, but we choose $g(e_t,T) = 1$, such that N is space-like, so N has the form of

$$N = k\, e_x + \, l\, e_y$$

And T has the form of:

$$T = e_t + e_x.$$

We know also $N \,\propto T'$ then $g(T,N) = 0$.
i.e., $g\left(e_t + e_x, k\, e_x + \, l\, e_y\right) = 0 = kg(e_x,e_x)$, then k must be zero. Then now N should be combined by:

$$N = (0,0,e_y)$$

Moreover, the binormal vector will be null. That means $g(B,B) = 0$
So, from supposition again, we have now, $g(T,T) = 0$ and $g(B,B) = g(T,N) = g(B,N) = 0$ while $g(T,B) = -1$

Now, one can try upon this setup to do as much of Frenet structure as possible:
Thus,

$$g(T,N) = 0 \;\Rightarrow T' = \;\kappa\, N$$
$$g(B,N) = 0 \;\Rightarrow B' = -\tau\, N$$

So, N'should be combining by:

$$N' = \propto T + \beta\, B$$

Therefore:

$$g(N',B) = \eta . g(T,B) = -\eta \;\text{ and } g\,(N',B) = - \;\; g\,(N,B') = -(-\tau)$$

Then: $\eta = -\tau$
also,

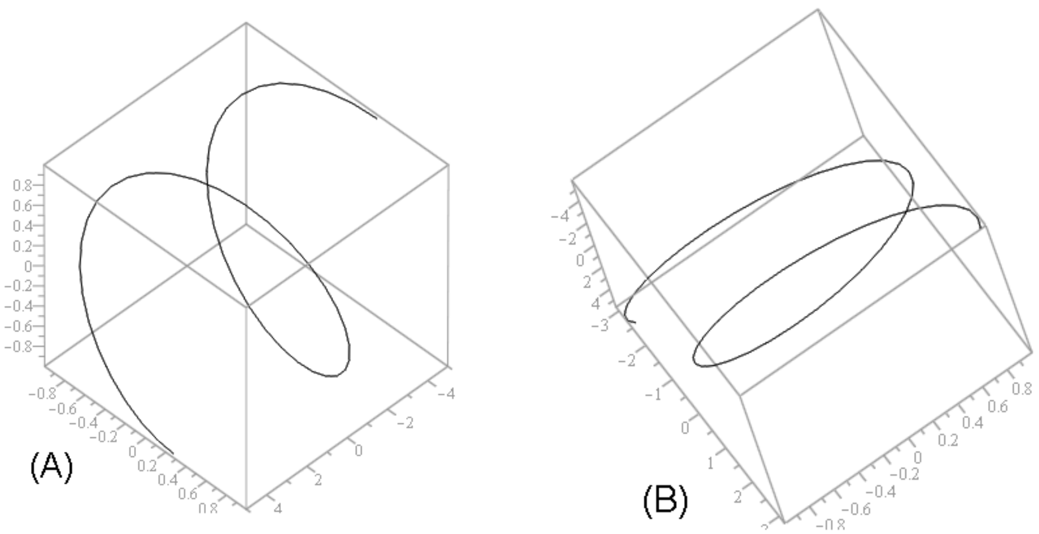
$$g(N',T) = \beta g(T,B) = -\,\beta . Also\; g(T',N) = -g(T,N') = -(\kappa)$$

Then: $\beta = \kappa$
Therefore, the Frenet Formula in matrix form for this case appears as:

$$\begin{pmatrix} \mathrm{T}'\,(\mathrm{s}) \\ N'(s) \\ \mathrm{B}'\,(\mathrm{s}) \end{pmatrix} = \begin{pmatrix} \theta & \kappa\,(\mathrm{s}) & \theta \\ -\,\tau(\mathrm{s}) & \theta & \kappa\,(\mathrm{s}) \\ \theta & -\,\tau(\mathrm{s}) & \theta \end{pmatrix} \begin{pmatrix} \mathrm{T}\,(\mathrm{s}) \\ N(s) \\ \mathrm{N(s)} \end{pmatrix}.$$

As a result, this formula is also interesting as a difference from all previous ones [11,12].

Figure 1. Figure A explains the null curve, and figure B show curve with Lorentz transformation, we can conclude that, the curve is different look and properties with Lorentz transforms.



(**a**) (**b**)

Discussion

An explicit example is provided below, this to demonstrate our supposition of this characterization of null curves in Minkowski spaces.
Consider the following Null curve in $\boldsymbol{M}^{2,1}$ formed by (t,x,y)defined as:

$$C(s) = (s, \cos\,(\mathrm{s}), \sin(s))$$

Differentiating of the above equation gives:

$$C'(s) = (1, -\sin(s)\;\, ,\cos(s))$$

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The dor (scalar) product of this curve is $g\ (C', C') = 0$. Then C is null (light) on curve. Though, we fester provide tangent of C as a vector:

$$T(s) = (1, -\sin(s)\ , \cos(s))$$

Then the principle normal vector

$$N(s) = (0, -\cos(s)\ , -\sin(s)\ )$$

Also $\kappa = |T'| = 1$.

Now let's write the binormal vector B,

$$B(s) = T(s) \times N(s) = (-1, -\sin\ (s)\ , \cos(s)\ )$$

To ensure the idea the operation × her is the cross in the sense of minkowski spacetime. Now differentiation of B for the function of torsion τ then

$$B'(s) = (0, -\cos(s)\ , -\sin(s)\ )$$
$$B'(s) = \ \tau\, N = (0, -\cos(s)\ , -\sin(s)\ )$$

So, $\tau = 1$.

We have now both curvature and torsion are equals and it is 1, which so-called helix of cylinder see figure A.

Following to our motivation, we are interested in studying this curve deeply. So, we applied the Lorentz transformation on this null curve. That to ensure whether the curve is still under same properties of null or not.

So,

$$C(s) = (s, \cos s\ , \sin s)$$

The Lorentz transformation (as known) of this curve is:

$$\begin{pmatrix} \tilde{t} \\ \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 \\ -v\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s \\ \cos s \\ \sin s \end{pmatrix} = \begin{pmatrix} \gamma s - v\gamma \cos s \\ \gamma \cos s - v\gamma s \\ \sin s \end{pmatrix}.$$

So, without loss of generality, by using the boost $v = \frac{1}{2}$ that is $\gamma = \frac{2}{\sqrt{3}}$. Then it's easy to check that the curve has been changed, and it is not null anymore. However, we will take a 2D projection of the curve $\tilde{C}$, that is to discuss how the two curves related. In other word to view the curvature of the 2D curve upon the Lorentz transforms. Although, the 2D projection of the original curve is a 2D parameterized curve, with curvature κ. But here the task is viewing how the two curves related.[13]

So,

$$\widetilde{C_2} = (\gamma \cos\ s - v\ \gamma s, \sin s\ )$$
$$\widetilde{C_2}' = (-\gamma \sin s - v\gamma, \cos s)$$
$$\widetilde{C_2}'' = (-\gamma \cos s\ , -\sin s\ )$$

Here, lets compute the curvature $\tilde{\kappa}$ of the 2D curve by using the formulae (see [3],[1])

$$\tilde{\kappa} = \frac{\det\,(\widetilde{C_2}', \widetilde{C_2}'')}{|\widetilde{C_2}'|^3}$$

Therefore,

$$\tilde{\kappa} = \frac{\gamma(1 + v \sin s)}{[\gamma^2(-\sin s - v)^2 + cos^2 s]^{3/2}}$$

The answer; there is no relation between two curves with 2D projection. They are different.

Finally, lets take the general case of the example above, and try to work out as much as the curvature$\tilde{\kappa}$.

If $C(s) = \big(s, x(s), y(s)\big)$ and, $|C'| = 0$, with $\big(x(s), y(s)\big)$ in $t = 0$ is arc-length parameterized.

So, $\kappa = \sqrt{x''(s)^2 + y''(s)^2} = |T'|$

Thus, the Lorentz transformation of $C(s)$ is given by:

$$\tilde{C} = (s\gamma - v\gamma x(s), \gamma x(s) - v\gamma s, y(s))$$

And, now project in 2D form:

$$\widetilde{C_2} = (\gamma x(s) - v\gamma s, y(s))$$

Then,

$$\widetilde{C_2}' = (\gamma x(s)' - v\gamma, y(s)')$$

Therefore,

$$\left|\widetilde{C_2}\right| = \sqrt{\gamma^2(x(s)' - v)^2 + y(s)'}$$

And,

$$\widetilde{C_2''} = (\gamma x(s)'', y(s)'')$$

Hence,

$\widetilde{C_2}' \times \widetilde{C_2}'' = \gamma[\kappa - vy(s)'']$,

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and then,

$$\tilde{\kappa} = \frac{\gamma[\kappa - vy(s)'']}{[\gamma^2(x(s)' - v)^2 + y(s)']^{3/2}}$$

This is a mess. There is no relation. Although; this is the general form.

Conclusion

To sum up. In this article we explored three different cases of moving time-like along Frenet trihedron. Its obtained-on difference of curvature place and changing of torsion place of the curve from the usual in $E^3$and from all cases. Moreover, the main propose of Light-like case has an analogue. Also, it is different to previous cases. The example shows that the curve is entirely different behaviour and properties over Lorentz transformation. And the 2D projection does give a different curve.

The future work. At this stage, there is a work done in 2010 to discover the bishop frames in Minkowski space. We are interested in recovering the parallel frame of null curves of 3D Minkowski space.

Conflicts of Interest

There is no any type of conflict.

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المستخلص

تتناول هذه الورقة دراسة الإطار الفرياني للمنحنيات العدمية في فضاء مينكوفسكي ثلاثي الأبعاد M_1^2. يتم التركيز على خصائص الإطار الفرياني لهذه المنحنيات، والذي يشمل الاتجاهات الثلاثة: المماس، العادي، والمماس الثنائي، بالإضافة إلى المعادلات التفاضلية المرتبطة بها. يُعد هذا الموضوع مهمًا في الهندسة التفاضلية، خاصة في دراسة هندسة لورنتزية، حيث تضيف الطبيعة غير المحددة لمتري مينكوفسكي تعقيدًا إضافيًا مقارنة بالهندسة الإقليدية. تتضمن الدراسة استنتاج صيغ جديدة لمعادلات الإطار الفرياني في حالة المنحنيات العدمية. كما يتم تقديم مثال عملي معروض باستخدام برنامج "Maple" لتوضيح النتائج. المنحنيات العدمية في فضاء مينكوفسكي تظهر اختلافات جوهرية في السلوك والخصائص مقارنة بالفضاء الإقليدي. حيث ان تطبيق تحويلات لورنتز يؤدي إلى تغييرات في خصائص المنحنى، مما يدل على عدم الاستمرارية في هذه الحالات. الورقة تقدم رؤية جديدة لدراسة المنحنيات العدمية، مع الإشارة إلى العمل المستقبلي لتوسيع هذه الدراسة إلى أطر موازية في فضاء مينكوفسكي.

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