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Original article

The Mathematical Structure of the Space of Test Functions

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Abstract
This paper aims to provide a simple and clear presentation of the mathematical structure of the space of test functions, highlighting the essential definitions, illustrative examples, and construction methods using the concept of convolution. These functions form the foundation of distribution theory and weak derivatives.
Keywords. Test Functions, Generalized Functions, Distribution Theory, Weak Derivatives.

Introduction

Test functions play an important role in many fields such as mathematics, optimization, and computer science, where they are used as standard examples to test whether a certain property or definition is satisfied. In particular, test functions play a key role in defining the concept of weak derivatives (derivatives of functions that are not differentiable in the classical sense) $[1-6]$. Moreover, in distribution theory, test functions are used to define generalized functions $[4,7]$. A test function is a smooth function with compact support: 'smooth function' means that the function has continuous derivatives of all orders, and 'compact support' means that the function vanishes outside of some bounded set.

Many ordinary functions, such as polynomials, exponential functions, and trigonometric functions, satisfy the first condition (smoothness), but they do not vanish outside a bounded set. Therefore, in many cases, it is not easy to represent test functions directly. Instead, constructing test functions requires combining different types of functions. Exponential functions are very important in constructing test functions when they are used together with other functions.

One common method to construct test functions is the convolution, i.e., combining two functions to produce a new function with smoother properties. If we take a locally integrable function f and a test function φ, we can define their convolution as

$$(f*\varphi)(x)=\int_{\mathbb{R}^n} f(x-y)\varphi(y)dy.$$

Where the integral is always finite, since φ is smooth and vanishes outside a bounded set. The resulting function is smooth function even when f is not continuous. For more details about the convolution, see e.g., $[1,9]$.

This paper is organized as follows. In Section 2, we define the support of a function and give different examples in which the support is compact or not compact. We also define the smooth function provided with many direct and indirect examples. In Section 3, we define the notion of test function and their properties, provide many examples of them, and explain how to construct them using convolution, along with several theorems and their proofs.

Preliminaries

Definition 2.1: Support

The support of a function $f:\mathbb{R}^n\to\mathbb{R}$ is the closure of the set $\{x\in\mathbb{R}^n: f(x)\neq 0\}$, i.e.

$$\operatorname{supp} f=\overline{\{x\in\mathbb{R}^n: f(x)\neq 0\}}$$

The space of continuous functions with compact support in $\Omega\subset\mathbb{R}^n$ is denoted by $C_c(\Omega)$ or $C_0(\Omega)$.

Examples 2.2

1-Let the function $f(x)$ be defined as:

$$f(x)=\chi_{[0,1]}(x)=\begin{cases}1 & \text{if } x\in [0,1]\\ 0 & \text{if } x\notin [0,1]\end{cases}$$

Then,

$$\operatorname{supp} f(x)=\overline{\{x\in\mathbb{R}: f(x)\neq 0\}}=\overline{[0,1]}=[0,1]$$

Therefore, the function $f(x)$ has compact support, which is $[0,1]$, but,

$$f(x)\notin C_c(\Omega),$$

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because $f(x)$ is not continuous.

2- Let the function $f(x)$ be defined as:

$$f(x) = \sin x.$$

Then, $\operatorname{supp} f(x) = \overline{\{x \in \mathbb{R} : \sin x \neq 0\}} = \overline{\mathbb{R} \setminus \{n\pi : n \in \mathbb{Z}\}} = \mathbb{R}.$

Therefore, the function $f(x)$ does not have compact support, because the support of the function $f(x)$ is the set of all real numbers $\mathbb{R}$, which is an unbounded set. Hence,

$$f(x) \notin C_c(\Omega)$$

3- Let the function $f(x)$ be defined as:

$$f(x) = \begin{cases} \exp\left(\dfrac{1}{x^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

Then, $\operatorname{supp} f(x) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}} = \overline{(-1,1)} = [-1,1].$

Therefore, the function $f(x)$ has compact support, which is the interval $[-1,1]$, and it is also a continuous function. Hence,

$$f(x) \in C_c(\Omega).$$

Definition 2.3: Smooth function[11]

A function $\varphi: \mathbb{R}^n \to \mathbb{R}$ is said to be smooth or infinitely differentiable if its derivatives of all orders exist and are continuous.

The set of all smooth functions in $\Omega \subset \mathbb{R}^n$ is denoted by $C^\infty(\Omega)$ or C^∞.

An easy and direct examples of smooth functions are Polynomials, exponential functions, and trigonometric functions, such as $\sin x$ and $\cos x$, because they are infinitely differentiable.

The following example is useful and will be used in the construction of test functions below.

Example 2.4

For $x \in \mathbb{R}$, the function

$$f(x) = \begin{cases} e^{\frac{1}{x}}, & x < 0 \\ 0, & x \geq 0 \end{cases}$$

is a smooth function. Since the exponential function and the zero function are smooth functions, it is sufficient to prove that f is infinitely differentiable at zero.

By definition, $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} \begin{cases} \frac{e^{\frac{1}{h}}}{h}, & h < 0 \\ 0, & h \geq 0 \end{cases}$

This means that $\lim_{h\to 0^+} \frac{f(h)}{h} = 0$, and

$\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{e^{\frac{1}{h}}}{h} = \lim_{h\to 0^-} \frac{\frac{1}{h}}{e^{-\frac{1}{h}}} = 0$, where we rewrite the numerator and denominator in the last limit to be able to apply the L'Hôpital's rule.

Thus $\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{-1}{e^{-\frac{1}{h}}} = 0 = 0$. Hence, $f'(0) = 0$.

Similarly, we can show that $f^{(k)}(0) = 0$, $k = 2,3,4,\cdots$, by using the L'Hôpital's rule k times.

Space of Test Functions

Definition 3.1: Test Function

A function $\varphi: \mathbb{R}^n \to \mathbb{R}$ is said to be a test function if it is infinitely differentiable function with compact support.

The space of test functions on Ω, denoted by $D(\Omega)$ or $C_c^\infty(\Omega)$

Remark

For $\Omega \subset \mathbb{R}^n$ and $k = 1, 2 \ldots\ldots$we denote by

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i. $C(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi \text{ is continuous}\}$.
ii. $C^k(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi \text{ is differentiable up to order } k\}$.
iii. $C^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{R} : \varphi \text{ is infinitely differentiable}\}$
iv. $C_c^k(\Omega) = C^k(\Omega) \cap C_c(\Omega)$.
v. $C_c^\infty(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.
vi. $L^p(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is measurable}, \ \|u\|_{L^p(\Omega)} < \infty\}$,

Where $\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx\right)^{1/p} \quad (1 \le p < \infty)$.

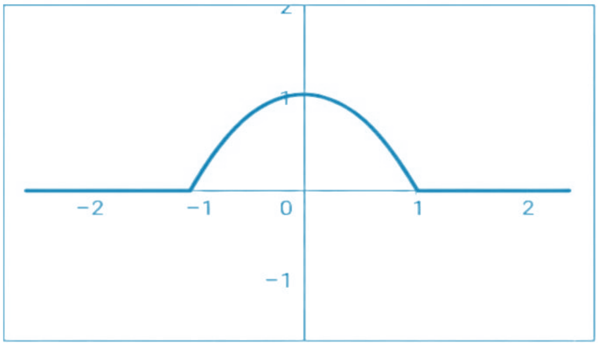
vii. $L_{loc}^p(\Omega) = \{u : \Omega \to \mathbb{R} : u \in L^p(V) \text{ for each } V \subset \bar{V} \subset \Omega\}$.

If $p = 1$, a function $f \in L^1(\Omega)$ is called integrable, and a function $g \in L_{loc}^p(\Omega)$ is called locally integrable.

Example 3.2: (of not Test Functions)

i- Let the function $u(x)$ defined as:

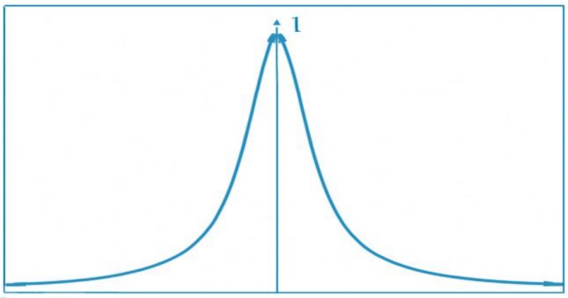
$$u(x) = \begin{cases} 1 - x^2 & x \in [-1, 1] \\ 0 & otherwise \end{cases}$$



The function $u(x)$ does not belong to the space of smooth functions because its derivative does not exist at 1 and -1, and therefore it does not belong to the space of test function, despite having compact support.

ii- Let the function $v(x)$ defined as:

$$v(x) = e^{-x}$$



The function $v(x)$ is differentiable in an infinite number of times, and therefore it belongs to the space of smooth functions. However, it does not have compact support because it never equals zero, and thus it does not belong to the space of test functions.

Example 3.3: (of Test Function)

First take $n = 1$**,** the real line, let φ be defined by

$$\varphi(x) = \begin{cases} \exp\left(\dfrac{1}{x^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \ge 1 \end{cases}$$

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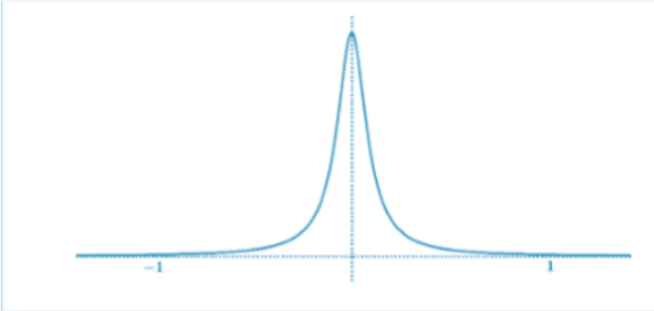
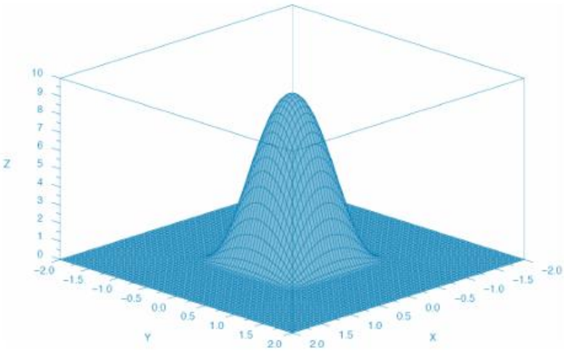


Then $\varphi \in C_c^\infty(\mathbb{R})$. To see this, notice that $\varphi(x) = f(x^2 - 1)$, for f in Example 2.4. Hence $\varphi \in C^\infty(\mathbb{R})$, and the supp $\varphi = \overline{\{x: \varphi(x) \neq 0\}} = \overline{\{x: |x| < 1\}} = |x| \leq 1$.

For arbitrary $n \geq 1$ we take, $\varphi(x) = \begin{cases} \exp\left(\frac{1}{\|x\|^2 - 1}\right) & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 1 \end{cases}$



Then $\varphi \in C_c^\infty(\mathbb{R}^n)$.

Properties of the Space of Test Functions

The following are some of the basic properties of the space of test functions.

1-*Linear Space*

The space of test functions is a linear space, i.e. if φ_1, φ_2 are a test function and $a, b \in \mathbb{R}$ then $(a\varphi_1 + b\varphi_2)$ is also a test function.

2-*Multiplication by a Smooth Function* [3]

If $\varphi(x)$ is a test function and $f(x)$ is an infinitely differentiable function, their product $f(x)\varphi(x)$ is also a test function.

3-*Basic Operations* [12]

Translation: If $\varphi(x)$ is a test function, the translated function $\varphi(x - a)$ is also a test function, where a is a real number.

Dilation: If $\varphi(x)$ is a test function, the function $\varphi(ax)$ is also a test function, where a is a real number.

4-Convergence[8]

A sequence of test functions $\{\varphi_n\}$ is said to converge to a test function φ in the space $C_c^\infty(\Omega)$ if,

i-there is a compact subset K of Ω such that supports of all the functions in the sequence $\{\varphi_n\}$ (and of φ) lie in K.

ii- φ_n and derivatives of φ_n of arbitrary order converge uniformly to φ and its derivatives.

Definition 3.4: Convolution

Let f, g be measurable functions on $\mathbb{R}^n$. The convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbb{R}^n} f(x - y)\, g(y) dy$$

For all $x \in \mathbb{R}^n$ such that the integral exists.

Some properties of convolution [1, 2, 10]

1-The convolution is defined for instance if $f \in L_{loc}^1$ and $\in C_c^\infty(\mathbb{R}^n)$. Then $f * \varphi = \varphi * f, f * \varphi \in C^\infty(\mathbb{R}^n)$ and $\partial^\alpha(f * \varphi) = f * \partial^\alpha \varphi$.

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Where $\alpha = (\alpha_1, \alpha_2, \dots\dots, \alpha_n) \in \mathbb{Z}^n$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$\partial^\alpha f = \frac{\partial^{|\alpha|} f}{\partial {x_1}^{\alpha_1} \cdots \partial {x_n}^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$$

2- For appropriate functions f and g one has

$$\operatorname{supp}(f * g) \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}\,.$$

3- If $f \in L^1_{loc}(\mathbb{R}^n)$ and $\in C_c(\mathbb{R}^n)$. Then $f * \varphi$ is continuous.

4-Let $f \in L^p_c(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n),\ u = 0 \text{ outside a compact of } \mathbb{R}^n\}$, For all $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have $\varphi * f \in C_c^\infty(\mathbb{R}^n)$.

Theorems on Test Functions

In this section we define $B_r(x) := \{y \in \mathbb{R}^n : \|y - x\| < r\}$.

Lemma 4.1: There exists a test function φ in $\mathbb{R}^n$ such that $\varphi \geq 0$, $\operatorname{supp} \varphi \subset B_1(0) = \{x : |x| \leq 1\}$ and $\int_{\mathbb{R}^n} \varphi\, dx = 1$.

Proof:

We have shown that the function φ in Example 3.3 is in the space $C_c^\infty(\mathbb{R}^n)$ and that $\operatorname{supp} \varphi = \{x : |x| \leq 1\}$, it remains to show that $\int_{\mathbb{R}^n} \varphi\, dx = 1$. To see this we can divide φ by $\int_{\mathbb{R}^n} \varphi\, dx$ and called again φ.

Remark: Let $\varepsilon > 0$, we define $\varphi_\varepsilon = \varepsilon^{-n} \varphi\left(\frac{x}{\varepsilon}\right), x \in \mathbb{R}^n$, where φ is the function defined in the previous Lemma. Notice that, we also have $\int_{\mathbb{R}^n} \varphi_\varepsilon\ dx = 1$. To see this we use change of variable $y = \frac{x}{\varepsilon}$ this implies that $dy = \varepsilon^{-n} dx$. Hence $\int_{\mathbb{R}^n} \varphi_\varepsilon\ dx = \int_{\mathbb{R}^n} \varphi\, dx = 1$.

Definition 4.2 [5]

For every $A, B \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we define

$$\operatorname{dist}(x, A) := \inf \{\|x - y\| : y \in A\}$$

$$\operatorname{dist}(A, B) := \inf \{\|x - y\| : x \in A, y \in B\}.$$

Theorem 4.3 [5] **:** Let f be locally integrable on $\mathbb{R}^n$(that is $f \in L^1_{\text{loc}}(\mathbb{R}^n)$) and $g \in C_c^k(\mathbb{R}^n)$. Then $f * g \in C^k(\mathbb{R}^n)$ and

$$\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g.$$

Proof:

Since $\partial^\alpha(f * g) = f * \partial^\alpha g$, we have $f * g \in C^k(\mathbb{R}^n)$. As for the support, notice that by the definition of convolution $f * g(x) = \int_{\mathbb{R}^n} f(x - y)\, g(y) dy$ and for $x \in \mathbb{R}^n$ such that $f * g(x) \neq 0$, it follows that, there exists at least $y \in \mathbb{R}^n$ satisfies $f(x - y) g(y) \neq 0$, which happens when $f(x - y) \neq 0$ and $g(y) \neq 0$ i.e., $x - y \in \operatorname{supp} f$ and $y \in \operatorname{supp} g$. Since $x = (x - y) + y$, we have $x \in \operatorname{supp} f + \operatorname{supp} g$. This shows that

$$\{x \in \mathbb{R}^n : f * g(x) \neq 0\} \subseteq \operatorname{supp} f + \operatorname{supp} g.$$

Since $\operatorname{supp} f$ is a closed set and $\operatorname{supp} g$ is a compact set, we have $\operatorname{supp} f + \operatorname{supp} g$ is a closed set. Now, we have

$\operatorname{supp}(f * g) = \overline{\{x \in \mathbb{R}^n : f * g(x) \neq 0\}} \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g} = \operatorname{supp} f + \operatorname{supp} g$, which ends the proof.

Theorem 4.4 [10] **:** If Ω is an open set in $\mathbb{R}^n$ and K is a compact subset of Ω, then there exists a function $\psi \in C_c^\infty(\Omega)$ with $0 \leq \psi \leq 1$ such that $\psi = 1$ in the neighborhood of K.

Proof:

Let $\varepsilon > 0$, be sufficiently small such that $\operatorname{dist}(K, \partial\Omega) \geq 4\varepsilon$. Let also $K_{2\varepsilon} := \{y \in \Omega :\ \|x - y\| \leq 2\varepsilon\}$ and $\chi_{K_{2\varepsilon}}$ be the characteristic function of $K_{2\varepsilon}$.

Now we define the function ψ as follows

$$\psi = \chi_{K_{2\varepsilon}} * \varphi_\varepsilon \in C_c^\infty(\mathrm{k}_{3\varepsilon}). \qquad (1)$$

C^∞regularity of ψ follows form properties of convolution and compactness, follows Theorem 4.3, since $\operatorname{supp}(\psi) \subset \operatorname{supp}(\chi_{K_{2\varepsilon}}) + \overline{B_\varepsilon(0)} \subset K_{3\varepsilon}$.

To show that $0 \leq \psi \leq 1$, we have from equation (1) that

$$0 \leq \psi(x) = \int_{\mathbb{R}^n} \chi_{K_{2\varepsilon}}(x - y) \varphi_\varepsilon(y) dy \leq \int_{\mathbb{R}^n} \varphi_\varepsilon(y) dy = 1$$

It remains to show that $\psi = 1$ in a neighborhood of K we just show that $1 - \psi$ vanishes in K_ε. Since,

$1 - \psi = 1 - \left(\chi_{K_{2\varepsilon}} * \varphi_\varepsilon\right) = 1 - \int_{\mathbb{R}^n} \chi_{K_{2\varepsilon}}(x - y) \varphi_\varepsilon(y) dy$

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$$= \int_{\mathbb{R}^n} \varphi_\varepsilon(y)dy - \int_{\mathbb{R}^n} \chi_{K_{2\varepsilon}}(x-y)\varphi_\varepsilon(y)dy$$

$$= \int_{\mathbb{R}^n} (1-\chi_{K_{2\varepsilon}})(x-y)\varphi_\varepsilon(y)dy = \left(1-\chi_{K_{2\varepsilon}}\right) * \varphi_\varepsilon.$$

Since $\operatorname{supp}(1-\psi) \subset \operatorname{supp}\left(1-\chi_{K_{2\varepsilon}}\right) + \overline{B_\varepsilon(0)} = \Omega \backslash K_{2\varepsilon} + B_\varepsilon(0) = \Omega \backslash K_\varepsilon$. It follows directly that $1-\psi = 0$ in K_ε, i.e., $\psi = 1$ in K_ε.

Theorem 4.5 [7] **:** Let $f \in C_c^k(\mathbb{R}^n)$, where $0 \le k < \infty$. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that

$$\varphi \ge 0, \quad \operatorname{supp} \varphi \subset \{|x| \le 1\}, \quad \int \varphi \, dx = 1,$$

Let ε be a positive real number, and put

$$f_\varepsilon(x) = (f * \varphi_\varepsilon)(x) = \varepsilon^{-n} \int f(y)\, \varphi\left(\frac{x-y}{\varepsilon}\right) dy. \qquad (2)$$

Then $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$, and the support of f_ε is contained in the ϵ-neighbourhood of the support of f; moreover, if $|\alpha| \le k$, then the $\partial^\alpha f_\varepsilon$ converge uniformly to $\partial^\alpha f$ as $\varepsilon \to 0$.

Proof:

C^∞regularity of f_ε follows form Theorem 4.3, since $\operatorname{supp}(f_\varepsilon) \subset \operatorname{supp}(f) + \overline{B_\varepsilon(0)}$, which is compact. To prove the convergence of $\partial^\alpha f_\varepsilon$, we use a change of variable $z = \frac{x-y}{\varepsilon}$ and write equation (2) as

$$f_\varepsilon(x) = \int f(x-\varepsilon z)\varphi(z)dz.$$

This implies that

$$|f_\varepsilon(x) - f(x)| = \left|\int f(x-\varepsilon z)\varphi(z)dz - f(x)\right| = \left|\int f(x-\varepsilon z)\varphi(z)dz - \int f(x)\varphi(z)dz\right| = \left|\int (f(x-\varepsilon z) - f(x))\varphi(z)dz\right|$$

$$\le \int |f(x-\varepsilon z) - f(x)|\, \varphi(z)dz \le \sup\{|f(x+y) - f(x)| : |y| \le \varepsilon\}.$$

The last inequality in the right-hand side tends to zero as $\varepsilon \to 0$. Hene $f_\varepsilon \to f$ uniformaly. Similarly, since

$$\partial^\alpha f_\varepsilon(x) = \partial^\alpha(f * \varphi_\varepsilon) = (\partial^\alpha f * \varphi_\varepsilon) = \int \partial^\alpha f(x-\varepsilon z)\, \varphi(z)dz,$$

we can show that $\partial^\alpha f_\varepsilon \to \partial^\alpha f$ uniformly as $\varepsilon \to 0$.

Conclusion

The paper highlights the importance of the space of test functions as a cornerstone in distribution theory and weak derivatives. These functions are used to define general concepts that go beyond the traditional limitations of differentiation. They also serve as a central tool in the analysis of partial differential equations. The paper explains how constructing these functions requires a deep understanding of smoothness and compact support, and it provides practical methods for their creation, such as convolution. This makes them valuable theoretically and practically in various fields of mathematics.

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