Original article

Effects of Exponential Mass Variation on Pendulum Damping and Resonance

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Abstract
The suggested paper presents a theoretical study to analyze the dynamical response of a simple pendulum carrying a time-dependent mass. The physical model introduced in this paper builds on the assumption that the mass of the hanging point particle varies exponentially with time. By using the extended Lagrangian formalism, which takes the reactive force (Meshchersky force) resulting from joining or losing mass into consideration, the equation of motion of the system was derived. The analysis shows that the effect of the exponential variation of the mass leads to the appearance of a physical term that behaves like a linear damping term in the equation of motion, defined as ρ hence, because of this damping coefficient, the total energy of the pendulum was exponentially decaying. The cases of I: null attached mass's velocity ($u = 0$) which leads to a free damped case, II: constant attached mass's velocity ($u = u_0$) which results in a steady shift in the position of the equilibrium point, III: variable attached mass's velocity ($u = u(t)$) which behaves like an external force term that can lead to a resonant amplification or change the spectrum of the dynamical response, were discussed. The analytical solutions, like the Laplace transform, Green's function, and spectral analysis, were obtained. The power delivered from the external driving force to the oscillator and the quality factor of the studied system were introduced.
Keywords. Exponential Mass Variation, Pendulum Damping, Resonance.

Introduction
Dynamically speaking, the variable mass system is a very classical, familiar problem in the field of classical mechanics. It is an old-fashioned problem with historical roots, and it has significant practical importance in oceanographic engineering, nutrition systems, and the interaction or integration of mechanical or celestial systems. Thanks to Meshchersky, who set up the mathematical cornerstone that interprets the reactive forces resulting from joining or losing masses from their original bodies, and his works in this field were a reference to formulate the equation of motion for a variable mass point particle [1].
While dealing with variable mass systems and when the Lagrangian approach is applied to these systems, a grand new physical terminology termed as "*reactive force*" or "*Metchersky's force*" appears in that context, and to provide the consistency of the Lagrangian approach with D'Alembert's principle of virtual work, it should be carried out carefully. A set of researchers presents new engineering applications and also seeks modified Lagrangian equations that can handle systems with time-dependent mass or position-dependent mass [2].
In Pendulua, the variation in the pendulum's mass clearly affects the dynamical response of the system, and this effect of mass variation becomes very notable since the addition or loss of mass influences the system's kinetic energy and moment of inertia and hence influences the resonance response, the quality factor, and the oscillation frequency of the system. Models for the variable mass pendulum in a series of former theoretical and experimental studies, including advanced analytical techniques like the wave analysis and the applications of Green's functions, numerical simulations, and the spectral analysis [3].
In this paper, the suggested procedure deals with a simple pendulum carrying a mass varying through time as $m_0 e^{\rho t}$ where ρ is the mass growth rate. By applying the extended Lagrangian formalism, which takes the reactive force into account due to the mass changes, the motion equation was derived. The small angle approximation, the Laplace transform for the analytical solution in the case of no velocity for the attached mass, i.e., u = 0, the Green's function method for the constant velocity case ($u = u_0$), and the spectral analysis method in the case of variable velocity u = u(t) [4].
The main aim of this paper is to show how the velocity of the attached mass u and the mass growth rate could affect the dynamical behavior of the simple pendulum, from an oscillatory damping to generating a new equilibrium position or even energy feedback. Also in this paper, an adequate analytical formula for evaluating the relaxation time, the frequency of the damped oscillations, and the Q-value. The results of the presented model can be applied in laboratory fields and as a starting point in designing sensors, depending on the dynamics of systems with variable mass [5].

Methods
Consider a simple physical oscillator (Pendulum) consisting of a point particle with time-dependent mass $m(t)$. The particle is suspended from a rigid, massless string of length l, whose upper end is fixed to a support. The system undergoes small planar angular oscillations. The focus of this study is the dynamical response of the pendulum when the suspended particle has a variable mass.

Received*: 05-08-2025* - ***Accepted****: 03-10-2025* - ***Published****: 12-10-2025*

The mass of the particle is assumed to vary exponentially with time according to the following formula:
$m(t) = m_0 e^{\rho t}$ (1)
where m_0 is the initial mass at $t = 0$, and ρ denotes the mass growth rate. To describe the dynamics, we derive the equation of motion for the pendulum with variable mass using the Lagrangian formalism. This formulation provides a consistent framework for analyzing how the changing mass modifies the oscillatory behavior of the system.
In the case of a variable mass system where the mass is varying with respect to the position, the exact form of the Lagrangian equation of motion is written as follows:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F(x, \dot{x}, t) - \frac{1}{2}\frac{dm}{dx}\dot{x} \quad (2)$$

where $F(q, \dot{q}, t)$ is the external applied force acting on the system. Actually, the expanded Lagrange equation of motion can be formulated for the case of a system of particles where the mass of the system is a function of the position, velocity, and time. Suppose a dynamical system consists of N particles, each of mass m_j positioned at r_i in a certain frame of reference, and let $p_j = m_i \frac{dr_i}{dt}$ be the linear momentum of each particle. If the considered system has gained or lost an amount of its mass at a velocity u with respect to the system, the extended Levi-Civita's expression of Newton's 2nd law of motion will be used in addition to D'Alembert's principle of virtual work [2]:

$$\sum F_i = \frac{dp_i}{dt} \rightarrow \sum F_i - \frac{dp_i}{dt} = 0 \quad (3)$$

Multiply both sides by $(\cdot\, \delta r_i)$, one gets:

$$\sum_i^n (F_i - \frac{dp_i}{dt}) \cdot \delta r_i = 0 \quad (4)$$

Here F_i represents the total force acting on the system. It is spliced into two individual forces, i.e. $F_i = \mathcal{F}_i + R_i$, where $\mathcal{F}_i$ is the total active force acting on each particle and $\mathcal{M}_i$ is known as the reactive force, which is proportional to the time rate of variation of the mass and to the velocity at which the mass is expelled or gained, i.e. $R_i = u_i \frac{dm_i}{dt}$. According to Russian technical literature, the reactive force R_i is termed as the "*Metchersky's force*" and it is a function of the relative velocity of the expelled or gained mass:

$$\mathcal{M}_i = (u_i - v_i)\frac{dm_i}{dt} \quad (5)$$

where u_i is the velocity of the incoming mass and v_i is the velocity of the swinging particle. Based on this interpretation, equation (4) can be rewritten as follows:

$$\sum_i^n [(\mathcal{F}_i + \mathcal{M}_i) - m_i \frac{dv_i}{dt}] \cdot \delta r_i = 0 \quad (6)$$

Therefore, the expanded Lagrange equation of motion will be:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = Q_i \quad (7)$$

Where Q_i represents the non-conservative generalized force, which consists of the external forces and the reactive force given as:

$$Q_i = \sum_i^n \left(\vec{\mathcal{F}_i} + (\vec{u_i} - \vec{v_i})\frac{dm_i}{dt}\right).\frac{\partial \vec{r_i}}{\partial \varphi} + \frac{1}{2}\sum_i^n \left[\frac{d}{dt}\left(\frac{\partial m_i}{\partial \dot{q}_i}\right) v_i^2 - \frac{\partial m_i}{\partial q_i} v_i^2\right] \quad (8)$$

Since the mass of the hanging particle changes only with time, then $\frac{\partial m_i}{\partial \dot{q}_i} = \frac{\partial m_i}{\partial q_i} = 0$. The force that acts on the point particle is its weight $\vec{\mathcal{F}_i} = -mg\hat{\jmath}$ and $\vec{U} = (\vec{u_i} - \vec{v_i}) = U\hat{\varphi}$ this is the tangential component of the relative velocity of the attached mass, such that $\hat{\varphi} = cos\varphi\hat{\imath} + sin\varphi\hat{\jmath}$ and the position vector for the variable mass object is $\vec{r_i} = lsin\varphi\hat{\imath} + lcos\varphi\hat{\jmath}$ and then $\frac{\partial \vec{r_i}}{\partial \varphi} = lcos\varphi\hat{\imath} + lsin\varphi\hat{\jmath}$. Hence, equation (8) becomes:

$$Q = \left(\mathcal{F}_i + \vec{U}\frac{dm_i}{dt}\right).\frac{\partial \vec{r_i}}{\partial \varphi} = -m(t)glsin\theta + \vec{U}l\frac{dm}{dt} \quad (8)$$

For the case of a varying mass pendulum, the kinetic energy of the system is given by:

$$T = \frac{1}{2}m(t)(\dot{x}^2 + \dot{y}^2) \quad (9)$$

Now, one has to define the Cartesian coordinates of the system (x, y) such that:

$$x = x(l, \varphi) = lsin\varphi \text{ and } y = y(l, \varphi) = l(1 - cos\varphi)$$

Then, using some calculus, one gets:

$$\dot{x} = \frac{\partial x}{\partial \varphi}\dot{\varphi} = l\dot{\varphi}\, cos\varphi \quad (10a)$$

$$\dot{y} = \frac{\partial y}{\partial \varphi}\dot{\varphi} = l\dot{\varphi} sin\varphi \quad (10b)$$

Squaring and adding (10a) and (10b), one obtains:

Received: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025

https://doi.org/10.54361/ajmas.258419

$\dot{x}^2 + \dot{y}^2 = l^2\dot{\varphi}^2$

Therefore, equation (9) becomes:

$$T = \frac{1}{2}m(t)l^2\dot{\varphi}^2 \qquad (11)$$

Hence, equation (7) becomes:

$$\frac{d}{dt}(m(t)l^2\dot{\varphi}) = -m(t)glsin\varphi + (u - v)l\frac{dm}{dt} \qquad (12)$$

Hence, with little algebra and using the small-angle approximation, one gets:

$$\frac{d^2\varphi}{dt^2} + \frac{\dot{m}(t)}{m(t)}\frac{d\varphi}{dt} + \omega_0^2\varphi = U\frac{\dot{m}(t)}{lm(t)} \qquad (13)$$

where $\omega_0^2 = \frac{g}{l}$ represents the simple pendulum's natural frequency and U is the relative velocity of the attached mass of the system. Equation (13) represents a second-order nonlinear and nonhomogeneous differential equation having initial conditions that describes the variations of the angular position of the pendulum with respect to time. Inserting $U = u - v$ into equation (13) and setting $v = l\frac{d\varphi}{dt}$ and also using the fact that $\frac{1}{m}\frac{dm}{dt} = \rho$, then arranging the equation, one gets:

$$\frac{d^2\varphi}{dt^2} + 2\rho\frac{d\varphi}{dt} + \omega_0^2\varphi = \frac{\rho}{l}u \qquad (14)$$

Case I: $u = 0$:

When the velocity of the attached mass in the pendulum's bob is null, that is $u = 0$. Therefore, the differential equation (14) reduces to the following form:

$$\frac{d^2\varphi}{dt^2} + 2\rho\frac{d\varphi}{dt} + \omega_0^2\varphi = 0 \qquad (15)$$

To solve this differential equation, one may apply the method of integral equations by transforming the equation from the differential form into an integral form [7], i.e.:

$$\varphi(t) = f(t) + \int_0^t W(x,t)\varphi(x)dx \qquad (16)$$

where $f(t) = (1 + 2\rho t)\varphi_0$ and $W(x,t) = (x - t)\omega_0^2 - 2\rho$. To solve the integral equation (15), one may use the Laplace transform method, i.e.:

$$\mathcal{L}[\varphi(t)] = \mathcal{L}[f(t)] + \mathcal{L}[\int_0^t W(x,t)\varphi(x)dx]$$

Let $\varphi(s) = \mathcal{L}[\varphi(t)]$and $F(s) = \mathcal{L}[f(t)]$. The Laplace transform for the second term is somehow different, such that:

$$\mathcal{L}[\int_0^t \big((x - t)\omega_0^2 - 2\rho\big)\varphi(x)dx] = (-\,\omega_0^2\mathcal{L}[t] + \mathcal{L}[2\rho])\varphi(s) = -(\frac{\omega_0^2}{s^2} + \frac{2\rho}{s})\,\varphi(s)$$

and

$$F(s) = \mathcal{L}[(1 + 2\rho t)\varphi_0\,] = (\frac{2\rho}{s^2} + \frac{1}{s})\varphi_0$$

where the Laplace transform is defined as:

$$\mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt$$

Then, after some mathematical manipulation, one gets:

$$\varphi(s) = \frac{F(s)}{1 + \frac{\omega_0^2}{s^2} + \frac{2\rho}{s}} = \frac{\frac{2\rho + s}{s^2}\varphi_0}{\frac{s^2 + 2\rho s + \omega_0^2}{s^2}} = \frac{(2\rho + s)\varphi_0}{s^2 + 2\rho s + \omega_0^2}$$

Then substituting $\varphi(s)$ by $\mathcal{L}[\varphi(t)]$*One* gets:

$$\mathcal{L}[\varphi(t)] = \frac{(2\rho + s)\varphi_0}{s^2 + 2\rho s + \omega_0^2}$$

By taking the inverse Laplace transform for both sides, one obtains:

$$\varphi(t) = \mathcal{L}^{-1}[\frac{(2\rho + s)\varphi_0}{s^2 + 2\rho s + \omega_0^2}]$$

To solve the above equation, one may use the fractional decomposition method, i.e.:

$$\frac{(2\rho + s)\varphi_0}{s^2 + 2\rho s + \omega_0^2} = \frac{A}{s - \alpha} + \frac{B}{s + \beta}$$

where $\alpha\ and\ \beta$ are the solutions of the dominator equations such that:

$$\alpha = -\rho + \sqrt{\rho^2 - \omega_0^2}\ and\ \beta = \rho + \sqrt{\rho^2 - \omega_0^2}$$

Therefore, after some algebra, one finds:

$A + B = \varphi_0$ and $\beta A - \alpha B = 2\rho\varphi_0$

Solving for A and B, one obtains:

***Received**: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025*

https://doi.org/10.54361/ajmas.258419

$A = \frac{\sqrt{\rho^2-\omega_0^2}-\rho}{2\sqrt{\rho^2-\omega_0^2}}\varphi_0$ and $B = \frac{\sqrt{\rho^2-\omega_0^2}+\rho}{2\sqrt{\rho^2-\omega_0^2}}\varphi_0$

Hence, the solution for the above equation is:

$$\varphi(t) = \frac{\sqrt{\rho^2-\omega_0^2}+\rho}{2\sqrt{\rho^2-\omega_0^2}}\varphi_0\,\mathcal{L}^{-1}\left[\frac{1}{s-\alpha}\right] + \frac{\rho-\sqrt{\rho^2-\omega_0^2}}{2\sqrt{\rho^2-\omega_0^2}}\varphi_0\,\mathcal{L}^{-1}\left[\frac{1}{s+\beta}\right] =$$

$$\frac{\sqrt{\rho^2-\omega_0^2}+\rho}{2\sqrt{\rho^2-\omega_0^2}}\varphi_0\,e^{\alpha t} + \frac{\rho-\sqrt{\rho^2-\omega_0^2}}{2\sqrt{\rho^2-\omega_0^2}}\varphi_0\,e^{\beta t}$$

Therefore, after doing some algebra, one obtains:

$$\varphi(t) = \varphi_0\,e^{-\rho t}\left[\frac{\rho}{\sqrt{\rho^2-\omega_0^2}}\left(\frac{e^{\sqrt{\rho^2-\omega_0^2}t}-e^{-\sqrt{\rho^2-\omega_0^2}t}}{2}\right)+\frac{e^{\sqrt{\rho^2-\omega_0^2}t}+e^{-\sqrt{\rho^2-\omega_0^2}t}}{2}\right] \quad (17)$$

Equation (17) represents the solution for the differential equation (14) by using the integral method. According to the sign of the quantity inside the radical, there will be three different states:

State I: When ($\rho^2 > \omega_0^2$) the quantity inside the radical is real and positive, *i.e.*$\sqrt{\rho^2-\omega_0^2} > 0$. In this case, the displacement function becomes*:*

$$\varphi(t) = \varphi_0 e^{-\rho t}[\cosh\left(\sqrt{\rho^2-\omega_0^2}.t\right) + \frac{\rho}{\sqrt{\rho^2-\omega_0^2}}\sinh(\sqrt{\rho^2-\omega_0^2}.t)] \quad (18)$$

Based on equation (18), it is clear that the behavior of the oscillator due to the presence of $\cosh\left(\sqrt{\rho^2-\omega_0^2}.t\right)$ and $\sinh\left(\sqrt{\rho^2-\omega_0^2}.t\right)$ terms of the equation, which are not periodic time functions, are not vibratory in nature. As time passes, the amplitude of oscillations of the damped oscillator will increase. This indicates that when the oscillator is shifted for the first time away from its equilibrium position, then, due to the restoring force of gravity, the oscillator will go far away from the relaxed position without executing any oscillations. The behavior of the oscillator in this case is called the positive heavy-damped oscillations. Therefore, if the damping factor ρ is as large as possible, and for this damping ρ has to be large*;* the oscillation is said to be heavily damped.

State II: When ($\rho^2-\omega_0^2 = \varepsilon^2 \ll 1$), the quantity ($\sqrt{\rho^2-\omega_0^2} = \varepsilon \ll 1$). In this case, equation (16) reveals that the angular displacement will be undefined. When the quantity ($\sqrt{\rho^2-\omega_0^2}$) approaches to zero, i.e. $\sqrt{\rho^2-\omega_0^2} \ll 1$, then the fact that ($e^x \cong 1 + x\ \forall\ x \ll 1$) will be used to treat this problem. Thus, the equation of displacement becomes:

$$\varphi(t) = \varphi_0 e^{-\rho t}[\frac{1+\sqrt{\rho^2-\omega_0^2}t+1-\sqrt{\rho^2-\omega_0^2}t}{2} + \frac{\rho}{\sqrt{\rho^2-\omega_0^2}}\frac{1+\sqrt{\rho^2-\omega_0^2}t-1+\sqrt{\rho^2-\omega_0^2}t}{2}]$$

Therefore:

$$\varphi(t) = \varphi_0 e^{-\rho t}(1+\rho t) = \varphi_0(1+\rho t)e^{-\rho t} \quad (19)$$

It can be seen from equation (19) that the behavior of the oscillator is not oscillatory, and the amplitude of the oscillations tends to a very high level in a very short time. In such a case, the reaction of the oscillator is called the negative critically damped oscillation*.*

State III: When ($\rho^2-\omega_0^2 < 0$) the quantity ($\sqrt{\rho^2-\omega_0^2}$) becomes a negative quantity, i.e. $\sqrt{\rho^2-\omega_0^2} = i\sqrt{\omega_0^2-\rho^2}$, then the angular displacement function will be*:*

$$\varphi(t) = \varphi_0 e^{-\rho t}[\frac{e^{i\sqrt{\rho^2-\omega_0^2}.t}+e^{-i\sqrt{\rho^2-\omega_0^2}t}}{2} + \frac{\rho}{\sqrt{\rho^2-\omega_0^2}}\frac{e^{i\sqrt{\rho^2-\omega_0^2}.t}-e^{-i\sqrt{\rho^2-\omega_0^2}t})}{2i}]$$

Using Euler's formula, one gets*:*

$$\varphi(t) = \varphi_0 e^{-\rho t}[\cos\left(\sqrt{\omega_0^2-\rho^2 t}\right) + \frac{\rho}{\sqrt{\omega_0^2-\rho^2}}\sin\left(\sqrt{\omega_0^2-\rho^2 t}\right)]$$

Using some trigonometric identities, one has:

$$\varphi(t) = \varphi_0 e^{-\rho t} * A\cos(\sqrt{\omega_0^2-\rho^2 t}-\delta)$$

where $A = \frac{\omega_0}{\omega}$ and $\delta = \tan^{-1}(\frac{\rho}{\omega})$. Therefore:

$$\varphi_{u=0}(t) = Ze^{-\rho t}\cos\left(\sqrt{\omega_0^2-\rho^2}t-\delta\right) \quad (20)$$

where $Z = A\varphi_0$ [8].

The case of the under-damped oscillation will be the core of the study in this work.

- ***Exponential envelope/time constant***

  The relaxation time constant is defined as the reciprocal of the damping coefficient, i.e. $\tau = \frac{1}{\rho}$. Thus at $\rho = 0.01$ leads $\tau = 100\ s$ (slow decay), while at $\rho = 0.04$ leads to $\tau = 25\ s$ (faster decay). This matches the

Received: *05-08-2025* - ***Accepted***: *03-10-2025* - ***Published***: *12-10-2025*

https://doi.org/10.54361/ajmas.258419

panels. When $\rho = 0.01$ trace remains appreciable at $t = 100\,s$, whereas at $\rho = 0.04$ The trace is essentially extinguished by that time.

- ***Damped oscillation frequency***
  The damped angular frequency is defined as $= \sqrt{\omega_0^2 - \rho^2}$. Because ρ is small relative to ω_0 in these examples, the change in period is modest; nevertheless, increasing ρ produces a slight reduction in oscillation frequency (longer period) visible as a subtle stretching of the oscillation in the higher-ρ panels.

- ***Energy and mechanism***
  Even with no external viscous friction, mass growth (positive$\dot{m}$) removes kinetic energy from the moving point particle via the mass-addition process; mathematically, this appears as the $2\rho\dot{\varphi}$ term (a dissipative term). Hence, higher ρ acts like stronger damping and accelerates the decay of mechanical energy.

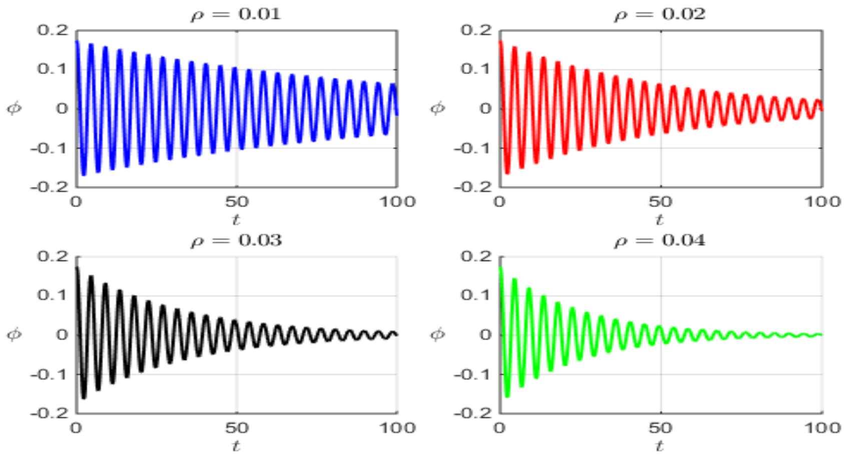

Fig.1. The change of the angular displacement as a function of time at a fixed ω_0 and different ρ when $u = 0$

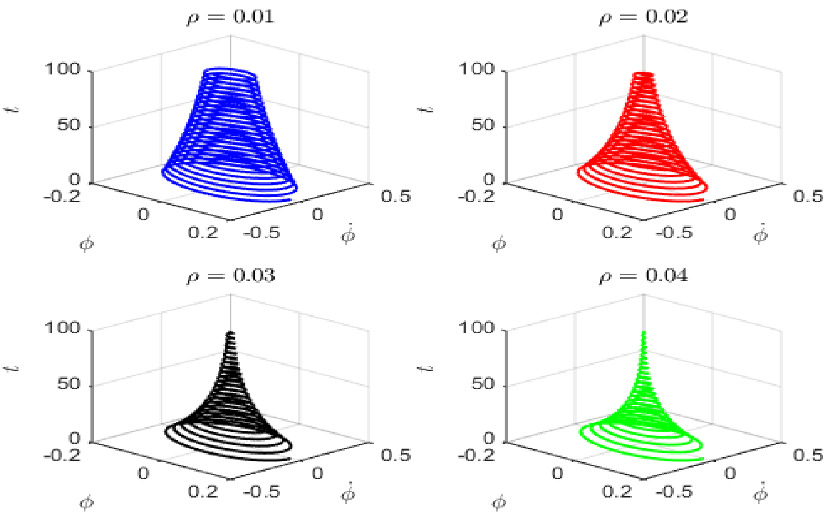

Fig.2. The 3D phase space of the angular displacement as a function of time at fixed ω_0 and different ρ when $u = 0$

Effect of u = 0 (attached-mass velocity) on the dynamical response:
When the velocity of the attached mass relative to the pendulum, u, is zero, the equation of motion reduces to a homogeneous damped oscillator. There is no external forcing from mass exchange, so the pendulum's oscillations decay monotonically (at an exponential envelope) toward the equilibrium position. In contrast, a nonzero u introduces a forcing term that can shift the equilibrium, sustain oscillations, or even inject energy depending on its sign and time dependence.

The Energy of the Variable Mass Pendulum
The total energy of a dynamical physical system under any physical circumstances is the sum of its kinetic energy and its potential energy, and for the case of a simple physical pendulum, it is given as:

$$E = T + U = \frac{1}{2}ml^2\dot{\varphi}^2 + mgl(1 - cos\varphi) = \frac{1}{2}ml^2\dot{\varphi}^2 + \frac{1}{2}mgl\varphi^2$$

Received: *05-08-2025* - ***Accepted***: *03-10-2025* - ***Published***: *12-10-2025*

https://doi.org/10.54361/ajmas.258419

Or

$E = \frac{1}{2}m(t)l^2(\dot{\varphi}^2 + \omega_0^2\varphi^2)$ (21)

Differentiate equation (19) with respect to time to obtain the expression for angular velocity as follows:

$\dot{\varphi}(t) = \frac{d\varphi}{dt} = -Ze^{-\rho t}(\rho\cos(\omega t - \delta) + \omega\sin(\omega t - \delta) = -Z\omega_0 e^{-\rho t}\cos(\omega t - \mu)$

where $\mu = \delta + \theta$ and $\theta = \tan^{-1}\frac{\omega}{\rho}$One may deduce that $\mu = \frac{\pi}{2}$ under the condition that $\dot{\varphi}(0) = 0$, therefore, the above equation will reduce to the following equation:

$\dot{\varphi}(t) = -Z\omega_0 e^{-\rho t}\sin\omega t$ (22)

Then, inserting equation (21) into equation (20), the total energy becomes:

$E(t) = \frac{1}{2}m_0 l^2 Z^2 {\omega_0}^2 e^{-\rho t}\left(sin^2\omega t + cos^2(\omega t - \delta)\right)$

Or

$E(t) = ke^{-\rho t}\,\psi(t)$ (23)

where $k = \frac{1}{2}m_0 l^2 Z^2 {\omega_0}^2$ *and* $\psi(t) = sin^2\omega t + cos^2(\omega t - \delta)$. From the expression of $\psi(t)$ one may notice that it is a periodic function, i.e.:

$\psi(t + T) = \psi(t)$

Therefore:

$E(t + T) = ke^{-\rho(t+T)}\psi(t + T) = ke^{-\rho t}\,\psi(t)e^{-\rho T} = E(t)e^{-\rho T}$ (24)

at t = 0, $E(T) = E(0)e^{-\rho T}$.

at t = T, $E(2T) = E(T)e^{-\rho T} = E(0)e^{-2\rho T}$.

at t = 2T, $E(3T) = E(2T)e^{-\rho T} = E(0)e^{-3\rho T}$.

at t = 3T, $E(4T) = E(3T)e^{-\rho T} = E(0)e^{-4\rho T}$.

at t = nT, $E((n + 1)T) = E(nT)e^{-\rho T} = E(0)e^{-n\rho T}$.

Therefore, the total energy of the system now becomes $E(t) = E(0)e^{-\rho t} = k\,\psi(0) = kcos^2(\delta)$. After doing some mathematical manipulation, equation (24) becomes:

$E_{u=0}(t) = E(0)e^{-\rho t} = kcos^2\delta e^{-\rho t} = k\left(1 - \left(\frac{\rho}{\omega_0}\right)^2\right)e^{-\rho t}$ (25)

Equation (25) states that the energy $E(t)$ decays exponentially in time with a rate constant $\rho > 0$. Differentiating gives the simple first-order differential law:

$\frac{dE(t)}{dt} = -\rho E(t)$

Hence, the instantaneous rate of energy loss is proportional to the current energy.

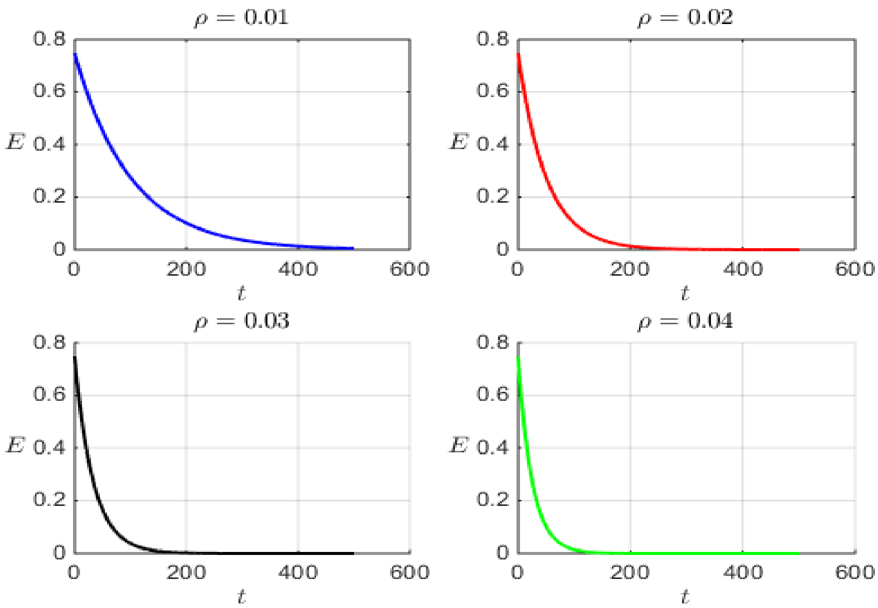

Fig.3. The energy of the variable mass pendulum as a function of time at different values of mass growth rate

This law describes purely exponential decay of the energy envelope; any fast oscillatory modulation of the instantaneous energy is carried separately by a T-periodic factor (if present) and does not affect the envelope. The formula implicitly assumes linear, time-homogeneous damping behavior (no parametric pumping or non-linear energy input) and that the normalization chosen makes$E(0) = cos^2\delta$. If mass, stiffness, or external forcing varies in time in ways that break these assumptions, the decay need not be purely exponential.

***Received**: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025*

The four panels display the unequivocal, exponentially fast extinction of the pendulum's mechanical energy as the mass-growth rate ρ increases. Each curve obeys equation (25):

Table 1. The values of the time constant at different values of the mass growth rate.

$\rho(s^{-1})$	$\tau(s)$
0.01	100
0.02	50
0.03	33.3
0.04	25

So the energy envelope decays purely exponentially with time constant $\tau = \frac{1}{\rho}$. As ρ rises from 0.01 to 0.04 the decay accelerates dramatically as shown in the table.1 and therefore the time to reduce the energy to a few percent of its initial value (roughly $t \approx 3\tau$) shrinks from ~300 to ~75 time units. The initial energy $E(0) = k\left(1-\left(\frac{\rho}{\omega_0}\right)^2\right)$ is only slightly reduced by ρ when $\rho \ll \omega_0$; the dominant effect visible in the plots is the envelope's exponential attenuation. All curves are strictly monotonic and show no long-term oscillatory persistence: increasing mass-growth rate does not supply sustaining energy; it suppresses the oscillations by steepening the exponential envelope.

Case II: $u = u_0$

When the velocity of the attached mass in the pendulum's bob is not zero, i.e., it is a constant. Then the differential equation (14) is going to be:

$$\frac{d^2\varphi}{dt^2} + 2\rho\frac{d\varphi}{dt} + \omega_0^2\varphi = \frac{\rho}{l}u_0 \qquad (26)$$

The differential equation (26) is going to be solved via the method of Green's function as follows [8]:

Let L be the linear operator of the system defined as $L = D_{tt} + 2\rho D_t + \omega_0^2$. The Green's function must satisfy the following two conditions:

$$\begin{cases} L\,[G(t)] = \delta(t) & \forall\, t > 0 \\ G(t) = 0 & \forall\, t < 0 \end{cases}$$

Hence, the function $\varphi(t)$ becomes:

$$\varphi(t) = \int_0^t G(t-\tau)F_0 d\tau \qquad (27)$$

If $t \neq 0 \rightarrow \ \delta(t) = 0 \rightarrow L\,G(t) = 0$, therefore $L\varphi = 0$:

$$D_{tt}\varphi + 2\rho D_t\varphi + \omega_0^2\varphi = 0$$

Therefore, the homogeneous solution will be:

$$\varphi_{1.2} = e^{-\rho t}\{cos\omega t, sin\omega t\}$$

where $\omega = \sqrt{\omega_0^2 - \rho^2}$. Insert causality by taking for $t > 0$, then:

$$G(t) = Ce^{-\rho t}sin\omega t \qquad (28)$$

Since $\sin(0) = 0$, then the continuity at $t = 0$ is automatic. The jump condition from integrating $L[\,G(t)] = \delta(t)$ at $t = 0$ is:

$$G'(0^+) - G'(0^-) = 1 \rightarrow G'(0^+) = 1 \rightarrow C = \frac{1}{\omega}$$

Therefore, equation (28) becomes:

$$G(t) = \frac{1}{\omega}e^{-\rho t}sin\omega t$$

Imposing the external force $F_0 = \rho\frac{u}{l}$ in equation (27), one gets:

$$\varphi(t) = \frac{F_0}{\omega}\int_0^t e^{-\rho p}sin\omega p dp = \frac{F_0}{\omega}S(t)$$

where $S(t) = \int_0^t e^{-\rho s}sin\omega s ds = \frac{1}{\rho^2+\omega^2}\left(1 - e^{-\rho t}\left(cos\omega t + \frac{\rho}{\omega}sin\omega t\right)\right)$

Therefore, the particular solution will be:

$$\varphi(t)_p = \frac{\rho u_0}{g}\left(1 - e^{-\rho t}\left(cos\omega t + \frac{\rho}{\omega}sin\omega t\right)\right) \qquad (29)$$

The general solution is the sum of the homogeneous and the particular parts:

$$\varphi_{u=u_0}(t) = \varphi_h(t) + \varphi_p(t) = \varphi_{u=0}(t) + \frac{\rho u_0}{g}\left(1 - e^{-\rho t}\left(cos\omega t + \frac{\rho}{\omega}sin\omega t\right)\right)$$

Doing some algebraic processes, finally, one gets:

$$\varphi_{u=u_0}(t) = \alpha\varphi_{u=0}(t) + \beta \qquad (30)$$

where $\varphi_{u=0}(t)$ is the angular position function for the case of $u = 0$, $\alpha = \left(1 - \frac{\rho u_0}{g\varphi_0}\right)$ and $\beta = \frac{\rho u_0}{g}$.

Received: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025

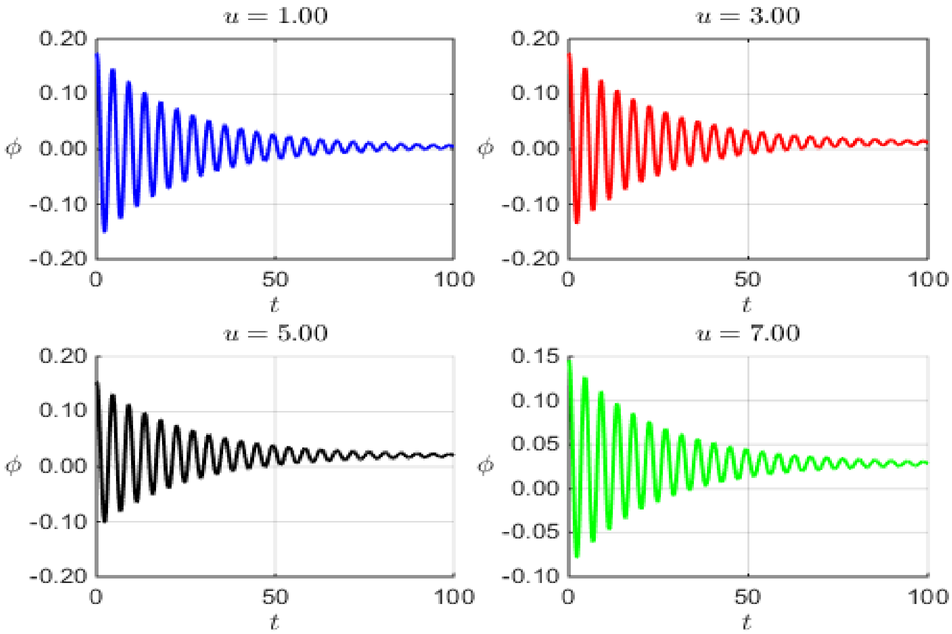

Fig.4. The change of the angular displacement as a function of time at fixed ω_0 and ρ when $u = [1, 3, 5, 7]$

Effect of $u = u_0$ on the System's Dynamical Response:

As u increase the final equilibrium (long-time mean of $\varphi(t)$) shifts upward: larger u produces larger positive steady offsets. In the (figure), this appears as the trace settling around progressively larger positive values when moving from u = 1 to u = 7.

The transient oscillation amplitude and the decay envelope are similar between panels (because ρ and ω_0 are fixed), but the instantaneous waveform near t = 0 and the approach to the nonzero offset change slightly due to the superposition of the homogeneous and particular solutions.

Quantitative/physical interpretation:

- ***Decay rate unchanged by u:*** The exponential envelope $\propto e^{-\rho t}$ is independent of the forcing magnitude u; hence, the transient lifetime $\tau = \frac{1}{\rho}$ is unchanged among panels. This explains why the oscillation amplitude decays at the same rate in all four subplots.
- ***Steady offset scales linearly with u:*** From $\varphi_p(t) = \frac{\rho u_0}{g}$ we expect the long-time mean to be proportional to u. The plot confirms that larger u produces proportionally larger static deflection.

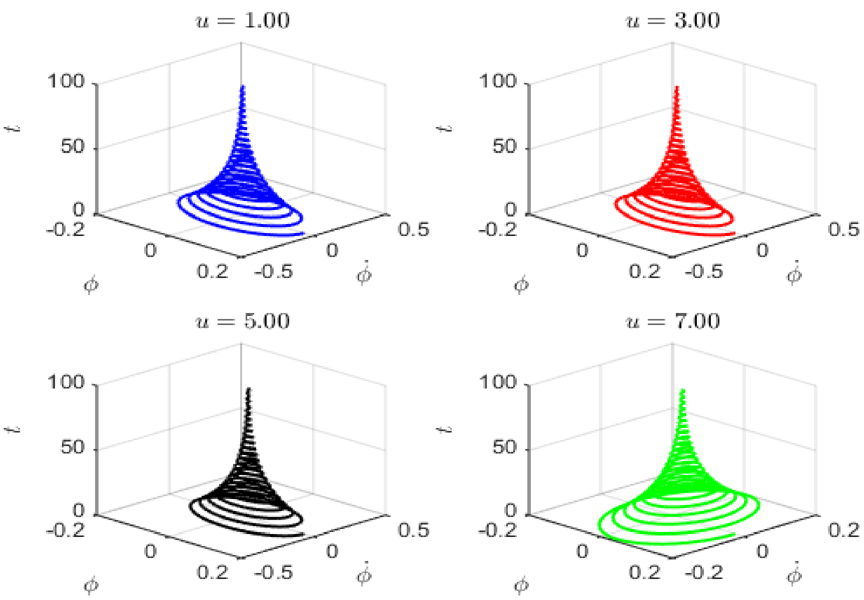

Fig.5. The 3D phase space of the angular displacement as a function of time at fixed ω_0 and ρ when $u = [1, 3, 5, 7]$

***Received**: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025*

https://doi.org/10.54361/ajmas.258419

- ***Oscillation frequency essentially constant:*** Because u is constant in time (no time-harmonic forcing), there is no new forcing frequency introduced; the oscillation period remains approximately $\frac{2\pi}{\omega}$. Only if u are time-dependent (e.g., harmonic) would you see forced oscillations at the forcing frequency and possible resonance effects.
- ***Energy & momentum viewpoint:*** Constant nonzero u corresponds to the attached mass joining the bob with a finite relative speed: momentum transfer produces a persistent bias (nonzero mean) in the bob position. The mass-growth term $2\rho\frac{d\varphi}{dt}$ still dissipates oscillatory kinetic energy, so oscillations decay while the mean shifts to φ_p.

The velocity of the system in this case is:

$$\dot{\varphi}_{u=u_0}(t) = \alpha\dot{\varphi}_{u=0}(t) \quad (31)$$

Therefore, inserting equations (28) and (29) into the energy equation (21), one gets:

$$E_{u=u_0}(t) = \alpha^2 E_{u=0}(t) + \frac{1}{2}m_0 l^2\omega_0^2\beta^2 e^{\rho t} + \alpha\beta m_0 l^2 Z cos(\omega t - \delta) \quad (32)$$

where $E_{u=0}(t) = k\left(1 - \left(\frac{\rho}{\omega_0}\right)^2\right)e^{-\rho t}$ is the total energy of the system when $u = 0$.

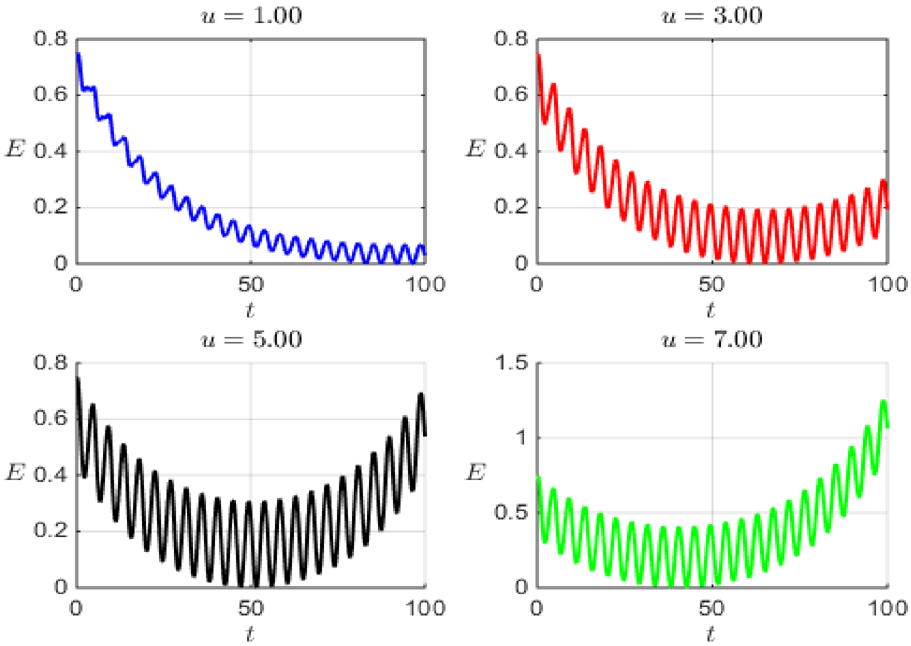

Fig.6. The variation of the total energy as a function of time at fixed ω_0 and ρ when $u = [1, 3, 5, 7] m/s$

(The figure) captures the transition from damping-dominated decay (u = 1) to fluctuation-dominated dynamics (u = 5,7) m/s**.** At small u, the pendulum energy primarily dissipates due to the exponential mass growth damping. As u increases, however, the coupling terms proportional to $\alpha\beta$ and $\beta^2 e^{\rho t}$ become more prominent, injecting oscillatory energy back into the system. This competition between dissipation (through ρ) and driving (through u) dictates the overall profile of E (t).

In essence, higher values of u act as an effective parametric drive, enriching the energy landscape with oscillations and delaying pure exponential decay. The figure beautifully illustrates how the attached mass velocity transforms the pendulum from a simple damped system into one exhibiting rich oscillatory energy dynamics.

Case III: $u = u_0 cos\Omega t$

When the velocity of the attached mass in the pendulum's bob varies with time. Then the differential equation (14) is going to be:

$$\frac{d^2\varphi}{dt^2} + 2\rho\frac{d\varphi}{dt} + \omega_0^2\varphi = F_0 cos\Omega t \quad (33)$$

where $F_0 = \frac{\rho}{l}u_0$. To solve the above non-homogeneous differential equation, one may use the following procedure [9].:

$$Z = \varphi + i\psi \quad (34)$$

Let Z be a solution for the following differential equation:

$$\frac{d^2Z}{dt^2} + 2\rho\frac{dZ}{dt} + \omega_0^2 Z = F_0 e^{i\Omega t} \quad (35)$$

Assume the solution as $Z = Z_0 e^{i\Omega t} \rightarrow \dot{Z} = i\Omega Z_0 e^{i\Omega t} \rightarrow \ddot{Z} = -\Omega^2 Z_0 e^{i\Omega t}$, substituting all of these in equation (31), one gets:

***Received**: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025*

https://doi.org/10.54361/ajmas.258419

$(\omega_0^2 - \Omega^2 + i2\rho\Omega)Z_0 e^{i\Omega t} = F_0 e^{i\Omega t}$

Finally, one obtains:

$$Z_0 = \frac{F_0}{(\omega_0^2 - \Omega^2 + i2\rho\Omega)} = \frac{F_0}{we^{i\theta}}$$

where $w = \sqrt{(\omega_0^2 - \Omega^2)^2 + 4(\rho\Omega)^2}$ and $\theta = \tan^{-1}\frac{2\rho\Omega}{\omega_0^2 - \Omega^2}$. Then:

$$Z = \frac{F_0}{we^{i\theta}} e^{i\Omega t} = \frac{F_0}{w} e^{i(\Omega t - \theta)} \qquad (36)$$

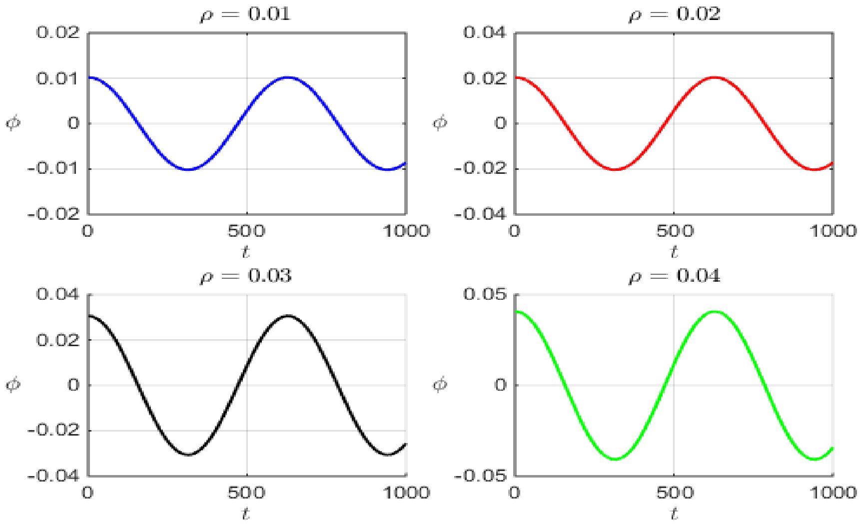

Fig.7. show variations in the angular displacement function as a function of time at different ρ and fixed ω_0 and Ω when $u = u_0 cos\Omega t$

Comparing equations (30) with equation (32) and taking the real part and ignoring the imaginary one, one gets:

$$\varphi(t, \Omega) = \frac{F_0}{w}\cos(\Omega t - \theta) = B(\Omega)\cos(\Omega t - \theta) \qquad (37)$$

where $B(\Omega) = \frac{F_0}{w}$. Equation (33) represents how the angular position of the variable mass pendulum changes over time as the velocity of the attached mass to the system varies with time.

(Figure 5) illustrates the variation of the angular displacement function $\varphi(t)$ as a function of time at fixed natural frequency ω_0 and excitation frequency Ω, when the velocity of the attached mass varies periodically. The external excitation originates from the time-varying velocity of the attached mass. When ρ is small, dissipation is weak, and the system exhibits a relatively strong oscillatory response. As ρ increases, energy losses due to damping dominate, leading to a progressive reduction in the amplitude of oscillations. Importantly, the frequency of oscillations remains essentially unchanged, as it is governed by the fixed parameters ω_0 and Ω.

The amplitude function $B(\Omega)$ is a function of the external excitation frequency Ω. The maximum value attained by the amplitude function $B(\Omega)$ can be given by the following condition, i.e. $\frac{dB}{d\Omega} = 0$ such that:

$$\frac{dB}{d\Omega} = -\frac{F_0}{2}((\omega_0^2 - \Omega^2)^2 + 4(\rho\Omega)^2)^{-\frac{3}{2}}.(-4\Omega(\omega_0^2 - \Omega^2) + 8\Omega\rho^2 \qquad (38)$$

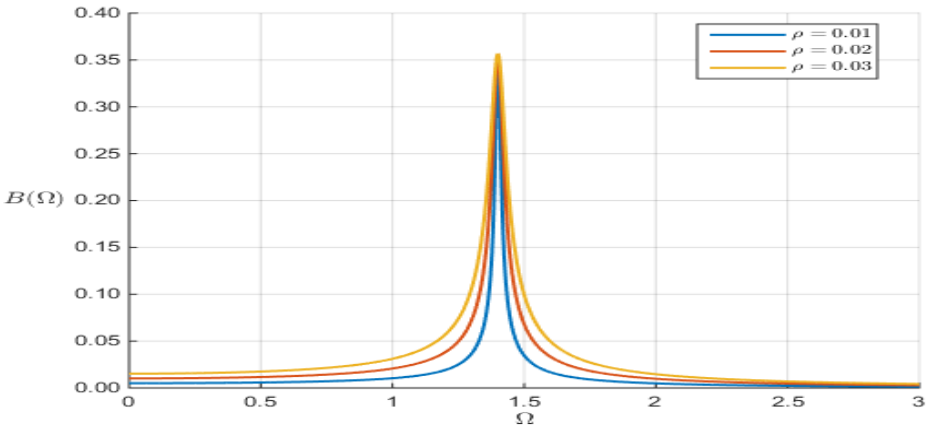

Fig.8. Variations in the amplitude function B (Ω) as a function of angular frequency Ω at different ρ and fixed ω_0 when $u = u_0 cos\Omega t$

***Received**: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025*

https://doi.org/10.54361/ajmas.258419

Equating equation (34) to zero and doing some algebra, one gets the critical external frequency at which the amplitude function attains its maximum:

$$\Omega^{*} = \sqrt{\omega_0^2 - 2\rho^2} \qquad (39)$$

At this critical value of the external frequency, the variable-mass pendulum system attains the resonance case. From (figure 8) one can notice that for $\Omega < \Omega^{*}$ the amplitude function $B(\Omega)$ is increasing for all values of the mass growth rate and $\Omega > \Omega^{*}$ it is decreasing for the same set of values for the mass growth rate. The velocity of the attached mass is given by $u = u_0 cos\Omega t$, provides the oscillatory driving force for the system. (Figure 8) shows that increasing the mass growth rate ρ suppresses and broadens the resonance response.

Power is transferred from the external driving source to the oscillator

To sustain steady-state oscillations, the external driving force must continually compensate for the energy dissipated during each cycle due to resistance. We now establish a key result: *in steady state, the amplitude and phase of a driven oscillator naturally adjust in such a way that the average power delivered by the driving force is exactly balanced by the power lost through frictional dissipation.* The instantaneous power P delivered at any moment is simply given by the product of the instantaneous driving force and the corresponding instantaneous velocity, i.e.:

$$P = -F_0 cos\Omega t \,.\, B' \sin(\Omega t - \theta) = -F_0 B' cos\Omega t \sin(\Omega t - \theta) \qquad (40)$$

where $B' = \frac{F_0}{\sqrt{(\frac{\omega_0^2}{\Omega} - \Omega)^2 + 4\rho^2}}$. The average power supplied by the outer driving force is defined as the total work done in one oscillation per one period of oscillation, in symbols:

$$\langle P \rangle = \frac{1}{T}\int_0^T P\, dt = \frac{-F_0 B'}{T}\int_0^T cos\Omega t \sin(\Omega t - \theta)\, dt$$

Expanding $\sin(\Omega t - \theta)$ and insert it inside the integral. One gets:

$$\langle P \rangle = \frac{-F_0 B'}{T}\int_0^T (cos\Omega t \sin\Omega t\, cos\theta - cos^2\Omega t\, cos\theta) dt$$

Or

$$\langle P \rangle = \frac{-F_0 B'}{T}[\int_0^T cos\Omega t \sin\Omega t\, cos\theta dt - \int_0^T cos^2\Omega t\, cos\theta\, dt \qquad (41)$$

Using the facts that $\int_0^T cos\Omega t \sin\Omega t\, cos\theta\, dt = 0$ *and* $\frac{1}{T}\int_0^T cos^2\Omega t\; dt = \frac{1}{2}$Then equation (41) becomes:

$$\langle P \rangle = \frac{{F_0}^2}{2I_m} cos\theta \qquad (42)$$

where $I_m = \sqrt{(\frac{\omega_0^2}{\Omega} - \Omega)^2 + 4\rho^2}$. The energy transported by the external driving force is not retained within the variable mass-pendulum system; instead, it is dissipated as work done in sustaining the motion of the system. The time rate variation of work done by the damping force is:

$$P = 2\rho\dot{\varphi}\dot{\varphi} = 2\rho\dot{\varphi}^2 = 2\rho\frac{{F_0}^2}{{I_m}^2} cos^2(\Omega t - \theta) \qquad (43)$$

Taking the average for equation (43) and using the above facts, one obtains:

$$\langle P \rangle = \rho\frac{{F_0}^2}{{I_m}^2}$$

It is known that the average power supplied by the outer driving force is equal to the average power dissipated by the damping term, i.e.:

$$\frac{{F_0}^2}{2I_m} cos\theta = \rho\frac{{F_0}^2}{{I_m}^2}$$

Solving for $cos\theta$One gets:

$$cos\theta = \frac{2\rho}{I_m}$$

When $\theta = 0$ then $I_m = 2\rho$, therefore, the average power for the system attains its maximum value, i.e.

$$\langle P \rangle_{max} = \frac{{F_0}^2}{4\rho} \qquad (43)$$

But $F_0 = \frac{\rho}{l} u_0$, hence equation (43) becomes:

$$\langle P \rangle_{max} = \frac{\rho}{4l} {u_0}^2$$

The Quality Value of the system

The quality factor or the Q-value of the oscillator is a physical parameter used to measure the rate of energy decay as time passes. In the case of a forced damped oscillator, the expression for the Q-value is given by:

Received: *05-08-2025 -* ***Accepted***: *03-10-2025 -* ***Published***: *12-10-2025*

$$Q = \frac{\omega_0}{\Omega_2 - \Omega_1} \quad (44)$$

where ω_1 and ω_2 are the frequencies at which $\langle P \rangle = \frac{1}{2}\langle P \rangle_{max}$, i.e.:

$$\rho \frac{{F_0}^2}{{I_m}^2} = \frac{{F_0}^2}{8\rho} \rightarrow {I_m}^2 = 8\rho^2 \rightarrow (\frac{\omega_0^2}{\Omega} - \Omega)^2 + 4\rho^2 = 8\rho^2 \rightarrow (\Omega - \frac{\omega_0^2}{\Omega})^2 = 4\rho^2$$

Then:

$$\Omega - \frac{\omega_0^2}{\Omega} = \pm 2\rho$$

If $\Omega_2 > \Omega_1$, hence:

$$\Omega_2 - \frac{\omega_0^2}{\Omega_2} = 2\rho$$

$$\Omega_1 - \frac{\omega_0^2}{\Omega_1} = -2\rho$$

Cancelling ω_0^2 from both equations, one obtains:

$$\Omega_2 - \Omega_1 = 2\rho \quad (45)$$

Inserting equation (45) into equation (44), one finally gets the Q-value for the system:

$$Q = \frac{\omega_0}{2\rho} \quad (46)$$

- ***Low ρ (0.01):*** The system is under-damped. It has a high Q-factor, leading to a tall, narrow peak. The system is very sensitive to frequencies near ω_0.
- ***Higher ρ (0.02, 0.03):*** The system becomes more over-damped. The Q-factor decreases, leading to a shorter, wider peak. The system is less sensitive to the exact driving frequency but has a more robust response over a range of frequencies.

Table 2. shows the Q-value of the system at different values of the mass growth rate in terms of the natural frequency.

$\rho(s^{-1})$	Q
0.01	$50\omega_0$
0.02	$25\omega_0$
0.03	$16.6\omega_0$

In essence, the mass growth rate ρ directly controls the amount of damping in the system, thereby determining how pronounced and selective the resonant response to the external drive $u = u_0 cos\Omega t$ will be.

Conclusion

In this paper, the suggested interpretations and analysis establish a clear, strong connection between the kinetics of the time variation of mass and pendulum dynamics: the exponential increase in the pendulum's inertia (mass) creates an analytically tractable damping behavior which contributes in the dissipation of the pendulum's energy and the sensitivity of the spectral response. On the other hand, the kinematics of the transferred mass controls the steady offsets and the responses of the external force. The combination of the effects specifies the stability, transient decay rates, and resonance amplification. The suggested model in this paper provides a very powerful tool and techniques for experimental or natural systems in which the mass is added or lost in a continuous way, and it clarifies the parameters (notably ρ and the spectral content of u (t) that are most influential for control, sensing, or instability avoidance.

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***Received**: 05-08-2025 - **Accepted**: 03-10-2025 - **Published**: 12-10-2025*