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Original article

Analytical Applications of Laplace Transforms in Solving Partial Differential Equations

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Abstract
The Laplace transform is a powerful mathematical tool for converting partial differential equations into simpler algebraic equations. This transformation method is widely used in engineering, physics, and applied mathematics to solve problems involving time-dependent processes. Laplace transforms are a popular tool for solving initial-value problems. In most undergraduate courses, the use of tables, special theorems, partial fractions, and convolution is the method taught for finding the inverse. In most problems involving partial differential equations. Consider a function *f(t)* such that *f(t)*= 0 for $t<0$. Then the Laplace integral is defined as $\mathcal{L}[f(t)] =$ *F(s)* $= \int_0^\infty f(t)e^{-st}dt$. The place transform is the most commonly employed analytic technique after separation of variables. One of the key elements of this method involves finding the Laplace transform of *u(x,t)* and its partial derivatives, where *u(x,t)* denotes the solution to the partial differential equations.the Laform of *u(x,t)* *is* d by the integral *U(x,s)* $= \int_0^\infty u(x,t)e^{-st}dt$.$\mathcal{L}[f(t)] =$ *F(s)* $= \int_0^\infty f(t)e^{-st}dt$. Clarify that Laplace transforms may be used to solve linear partial differential equations. And the solution of linear partial differential equations by Laplace transforms is the most commonly employed analytic technique after separation of variables. One of the key elements of this method involves finding the Laplace transform of *u(x,t)* and its partial derivatives, where *u(x,t)* denotes the solution to the partial differential equations.the Laplace transform of *u(x,t) is* defined by the integral *U(x,s)* $= \int_0^\infty u(x,t)e^{-st}dt$.
Keywords: Laplace transforms, Linear ordinary differential equations, First shifting theorem, Partial differential equations, Multivalued functions.

Introduction
This paper aims to clarify the theoretical foundations of the Laplace transform, discuss its key properties, and demonstrate its application in solving linear differential equations. Many physical processes in nature evolve within domains that may be considered infinite or semi-infinite in extent [1-5]. Consequently, the Laplace transform has proven to be a powerful analytical tool for addressing linear partial differential equations commonly encountered in engineering and the sciences [6-12]. The objective of this study is to illustrate the practical utility of Laplace transforms by establishing a unified conceptual starting point. To achieve this, the paper provides a structured review of Laplace transforms, ordinary differential equations, and relevant aspects of complex variable theory.

Preliminaries
Here, I recall the definitions of the Laplace transform and using the Laplace transform to solve partial differential equations.

Definition 1
Consider a function *f(t)* such that *f(t)* = 0 for $t<0$. Then the *Laplace integral*

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

defines the Laplace transform of *f(t)*, which I write $\mathcal{L}[f(t)]$ or *F(s)*. The Laplace transform of *f(t)* exists, for sufficiently large s, provided *f(t)* satisfies the following conditions:
f(t) = 0 for $t<0$,
f(t) is continuous or piecewise continuous in every interval,
$t^n|f(t)| < \infty$ *as* $t \to 0$ *for some number* n, *where* $n<1$,
$e^{-s_0 t}|f(t)| < \infty$ *as* $t \to \infty$ *for some number* s_0. *The quantity* s_0 is called the abscissa of convergence.

Example 1
Let us find the Laplace transform for the *Heavisine step function*:

$$H(t-a)= \begin{cases} 1, & t > a, \\ 0, & t < a. \end{cases}$$

The Heavisine step function is essentially a bookkeeping device that gives us the ability to"switch on" and "switch off" a given function. For example, if we want a function *f(t)* to become nonzero at time *t,* we represent this process by the product *f(t) H(t-a).*
From the definition of the Laplace transform**,**

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$$\mathcal{L}[H(t-a)] = \int_a^{\infty} e^{-st}dt = \frac{e^{-as}}{s}, \quad s>0$$

Example 2

The *Dirac delta function* or *impulse function*, often defined for computational purposes by

$$\delta(t) = \lim_{n\to\infty} \delta_n(t) = \lim_{n\to\infty} \begin{cases} n/2, & |t| < \frac{1}{2}, \\ 0, & |t| > \frac{1}{2}. \end{cases}$$

Plays an especially important role in transform methods because its Laplace transform is

$$\mathcal{L}[\delta(t-a)] = \int_0^{\infty} \delta(t-a)\, e^{-st}dt = \lim_{n\to\infty} \frac{n}{2} \int_{a-1/n}^{a+1/n} e^{-st}dt$$

Table 1: Some General Properties of Laplace Transforms with a > 0

No.	Property	Function, f(t)	Laplace Transform, F(s)
1	Linearity	$c_1 f(t) + c_2 g(t)$	$c_1 F(s) + c_2 G(s)$
2	Scaling	$f(t/a)/a$	$F(as)$
3	Multiplication by e^{bt}	$e^{bt} f(t)$	$F(s-b)$
4	Translation	$f(t-a)\, H(t-a)$	$e^{-as} F(s)$
5	Differentiation	$f^{(n)}(t)$	$s^{(n)}F(s) - s^{(n-1)}f(0) - f^{(n-1)}(0)$
6	Integration	$\int_0^t f(\tau)\, d\tau$	$F(s)/s$
7	Convolution	$\int_0^t f(t-\tau)g(\tau)\, d\tau$	$F(s)\, G(s)$

Definition 2

Consider now the transform of the function ￼￼￼ $e^{-at}f(t)$ where a is any real number￼, definition where a is any real number, then the definition

$$\mathcal{L}[e^{-at}f(t)] = \int_0^{\infty} e^{-st} f(t) e^{-at} f(t) dt \;\dots\dots\dots\dots\dots\dots\dots\dots\textbf{(1.1)}$$

Or

$$= \int_0^{\infty} e^{-(s+a)t} f(t)\, dt \;\dots\dots\dots\dots\dots\textbf{(1.2)}$$

$$\mathcal{L}[e^{-at}f(t)] = F(s+a) \quad \dots\dots\dots\dots\dots\dots\textbf{(1.3)}$$

Such that the function is known as the *first shifting theorem* and states that if $F(s)$ is the transform of $f(t)$ and a is a constant, then $F(s+a)$ is the transform of $e^{-at}f(t)$.

Example 3

Let us find the Laplace transform of $f(t) = e^{-at}sin(bt)$ .Because the Laplace transform of ￼ is *b/(ssin(bt)+b²),b/(s²+b²),*

$$\mathcal{L}[e^{-at}sin(bt)] = \frac{b}{(s+a)^2+b^2} \;\dots\dots\dots\dots\dots\dots\dots\dots\textbf{(1.4)}$$

where I have simply replaced s by $s+a$ in the transform for $sin(bt)$.

Example 4

Let us find inverse of the Laplace transform

$$F(s) = \frac{s+2}{s^2+6s+1} \;\dots\dots\dots\dots\dots\dots\dots\dots\textbf{(1.5)}$$

Rearranging terms,

$$F(s) = \frac{s+2}{s^2+6s+1} = \frac{s+2}{(s+3)^2-8}\dots\dots\dots\dots\dots\dots\dots\dots\textbf{(1.6)}$$

$$= \frac{s+3}{(s+3)^2-8} - \frac{1}{2\sqrt{2}}\frac{2\sqrt{2}}{(s+3)^2-8} \dots\dots\dots\dots\dots\dots\dots\dots \textbf{(1.7)}$$

Immediately, from the *first shifting theorem*,

$$f(t) = e^{-3t}cosh(2\sqrt{2}t) - \frac{e^{-3t}}{2\sqrt{2}} sinh(2\sqrt{2}t) \dots\dots\dots\dots\dots\dots\dots\dots \textbf{(1.8)}$$

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Definition 3

Consider the Laplace transform of $f(t-b)\ H(t-b)$, then the definition of the *second shifting theorem*: if $F(s)$ is the transform of ￼ then $F(s)$￼ is is the transform of $f(t)$, then $e^{-bs}F(s)$ is the transform of $f(t-b)\ H(t-b)$, where b is any real number and positive as

$$\mathcal{L}[f(t-b)H(t-b)] = \int_0^\infty f(t-b)\ H(t-b)e^{-st}dt = \int_b^\infty f(t-b)\, e^{-st}dt$$

$$= \int_0^\infty e^{-bs}\, e^{-sx}\, f(x)dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{1.9})$$

Or

$$\mathcal{L}[f(t-b)H(t-b)] \ = \ e^{-bs}\, F(s) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathbf{1.10})$$

Linear ordinary differential equations

Most analytic techniques for solving a partial differential equation involve reducing it to an ordinary differential equation or a set of ordinary differential equations that is hopefully easier to solve than the original partial differential equation.

From the vast number of possible ordinary differential equations, I focus on second-order equations. All of the following techniques extend to higher-order equations.

Consider the ordinary differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.1)$$

Where a, b, and c are real. For the moment, let us take $f(x) = 0$

Assuming a solution of the form $y(x) = Ae^{mx}$ and substituting into the Equation (2.1)

$$am^2 + bm + c = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.2)$$

This purely algebraic equation is the *characteristic or auxiliary equation*. Because equation (2) is quartic, there are either two real roots, or else a repeated real root, or else conjugate complex roots.

At this point, let us consider each case separately and state the solution. Any undergraduate book on ordinary differential equations will provide the details for obtaining these general solutions.

Case : Two distinct real roots m_1 and m_2,

$$y(x) = c_1e^{m_1x} + c_2e^{m_2x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.3)$$

Case ￼ : A repeated real root m_1,**II :** A repeated real root m_1**,**

$$y(x) = c_1e^{m_1x} + c_2xe^{m_1x}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.4)$$

Case : Conjugate complex roots $m_1 = p + qi$ and $m_2 = p - qi$,

$$y(x) = c_1e^{px}cos(qx) + c_2e^{px}sin(qx)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5)$$

Example 5

One of the most encountered differential equations is

$$\frac{d^2y}{dx^2} - m^2y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.6)$$

Where m is real and positive. Because there are two distinct roots, $m_{1,2} = \pm m$ the general solution is

$$y(x) = Ae^{mx} + Be^{-mx}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.7)$$

Although this solution is perfectly correct, it is most useful in a semi-infinite domain. For finite domains, such as $0 < x < L$. A little algebra shows that equation (7) also equals

$$y(x) = C\,cosh(mx) + D\,sinh(mx) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.8)$$

Where

$$cosh(mx) = \frac{1}{2}(\ e^{mx} + e^{-mx}\ ) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.9)$$

and

$$sinh(mx) = \frac{1}{2}(\ e^{mx} - e^{-mx}\ ) \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.10)$$

The advantage of using equation (8) follows from the fact sinh(0)=0 and cosh(0)=1

Example 6

Let us find the solution to the equation

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$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=cos^2(x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.11)$$

Our guess for a particular solution is then

$$y_p(x) = A+Bcos(2x)+Csin(2x)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.12)$$

Because $cos^2(x)=[1+cos(2x)]/2$ substituting equation (12) into equation (11) and equating coefficients for the constant, cosine, and sine terms, I find that A=$\frac{1}{2}$, B=-$\frac{3}{50}$, C=$\frac{2}{25}$.

The remaining task is to compute the arbitrary constants in the homogeneous solution. In this paper, I always have conditions at both ends of a given domain, even if one of these points is at infinity.

Now I illustrate the products used in solving these *boundary-value problems*.

Example 7

Solve the *boundary-value problem*

$$\frac{d^2y}{dx^2}-sy=-\frac{1}{s}, \quad y(0)=y(1)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.13)$$

Where s > 0. The general solution to equation (13) is

$$y(x)=A\ sinh(x\sqrt{s})+B\ cosh(x\sqrt{s})+\frac{1}{s^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.14)$$

I have chosen to use hyperbolic functions because the domain lies between *x=0 and x=1*. Now,

$$y\ (0)=B+\frac{1}{s^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.15)$$

and

$$y\ (1)=A\ sinh\ (\sqrt{s})+B\ cosh(\sqrt{s})+\frac{1}{s^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.16)$$

Solving for A and B,

$$A=\frac{cosh(\sqrt{s})-1}{s^2 sinh\ (\sqrt{s})} \quad \text{and} \quad B=-\frac{1}{s^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2.17)$$

Therefore,

$$y(x)=\frac{1-cosh(x\sqrt{s})}{s^2}+\frac{cosh(\sqrt{s})-1}{s^2 sinh\ (\sqrt{s})}sinh(x\sqrt{s}) \quad \ldots\ldots\ldots\ldots\ldots(2.18)$$

Problem

Solve the boundary-value problem

$$\frac{d^2y}{dx^2}-(a^2+s)y=0, y(-1)=\frac{1}{s}, y(1)=0$$

Where a and s are really positive.

Complex variables

Complex variables provide analytic tools for the evaluation of integrals with an ease that rarely occurs with real functions. The power of integration on the complex plane has its roots in the basic three C's: the Cauchy-Riemann equations, the Cauchy-Goursat theorem, and Cauchy's residue theorem. The Cauchy-Riemann equations have their origin in the definition of the derivative in the complex plane. Just as I have the concept of the function in real variables, where for a given value of, I can compute a corresponding value of $y=f(x)$.

I can define a complex function $w=f(z)$ where for a given value of $z=x+iy$ (i $=\sqrt{-1}$) I may compute $w=f(z)=u(x,y)+iv(x,y)$.

For $f'(z)$ to exist in some region, $u(x,y)$ and $v(x,y)$ *must* satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\ldots\ldots\ldots\ldots\ldots\ldots(3.1)$$

If u_x, u_y, v_x and v_y are continuous in some region surrounding a point z_0 and satisfy the equation (3.1) there, then $f(z)$ is *analytic* there. If a function is *analytic* everywhere in the complex plane, then it is an *entire* function.

Alternatively, if the function is analytic everywhere except at some isolated singularities, then it is *meromorphic*. Note that $f(z)$ must satisfy the Cauchy-Riemann equations in a region and not just at a point. For example, $f(z)=|z|$ satisfies the Cauchy-Riemann equations at z=0 and nowhere else. Consequently, this function is not analytic anywhere on the complex plane.

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Integration on the complex plane is more involved than in real, single variables because $dz = dx + i\,dy$. We must specify a path or contour as we integrate from one point to another. To see why, I introduce the following results:

Cauchy-Goursat theorem 1

If $f(z)$ is an *analytic function* at each point within and on a closed contour C, then is an *analytic function* at each point within and on a closed contour C, then

$\oint_C f(z)dz = 0.$

This theorem leads immediately to

The principle of deformation of contours

The value of a line integral of an analytic function around any simple closed contour remains unchanged if I deform the contour in such a manner that I do not pass over a point where *f(z)* is not analytic.

Consequently, I can evaluate difficult integrals by deforming the contour so that the actual evaluation is along a simpler contour or the computations are made easier, as seen (3.1).

Most integrations on the complex plane, however, are with meromorphic functions. The Next theorem involves these functions; it is

Cauchy's residue theorem 2

If f(z) is analytic inside a closed contour C (taken in the positive sense) except at points $z_1, z_2, \ldots, z_n$ where *f(z)* has singularities, then

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}\,[\,f(z);\ z_j\,] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.2)$$

where Res [*f(z)*; z_j] denotes the residue of *f(z)* for the singularity located at z_j.

The question now turns to what a residue is and how I compute it. The answer involves the nature of the singularity and an extension of the Taylor expansion, called *a Laurent expansion.*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_j)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_j)^{-n} \ldots\ldots\ldots\ldots\ldots\ldots (3.3)$$

For $0 < z - z_j < a <$. The first summation is merely the familiar Taylor expansion; the second summation involves a negative power of $z - z_j$ and gives the behaviour at the singularity. The *residue* equals the coefficient a_{-1}. and gives the behavior at the singularity. The *residue* equals the coefficient a_{-1}.

Turning to the nature of the singularity, there are three type:

Essential Singularity: Consider the function $f(z) = cos(1/z)$. Using the expansion for cosine,

$$cos\left(\frac{1}{z}\right) = 1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \frac{1}{6!\,z^6} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

For $0 < |z| < \infty$. Note that this series never truncates in the inverse powers of z. Essential singularities have *Laurent expansions* that have an infinite number of inverse powers of for $z - z_j$. *The value of the residue for* this essential singularity at z=0 is zero.

Removable Singularity: Consider the function $f(z) = sin(1/z)$. This function appears, at first blush, to have a Singularity at z=0. Upon applying the expansion for sine,

$$\left(\frac{sin\,(z)}{z}\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \frac{z^8}{9!} - \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.5)$$

For all z, I have no singularity at all. This is an example of a removable singularity; its residue is zero.

Pole of order n: Consider the function

$$f(Z) = \frac{1}{(Z-1)^3(Z+1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.6)$$

This function has two singularities: one at z =1 and the other at z =− 1.

I shall only consider the case z =1. After a little algebra,

$$f(Z) = \frac{1}{(Z-1)^3} \frac{1}{2+(Z-1)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.7)$$

$$f(Z) = \frac{1}{2} \frac{1}{(Z-1)^3} \frac{1}{1+(Z-1)/2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.8)$$

$$f(Z) = \frac{1}{2} \frac{1}{(Z-1)^3} \left[1 - \frac{(Z-1)}{2} + \frac{(Z-1)^2}{4} - \frac{(Z-1)^3}{8} + \ldots\right] \ldots\ldots\ldots\ldots\ldots (3.9)$$

$$f(Z) = \frac{1}{2} \frac{1}{(Z-1)^3} - \frac{1}{4(Z-1)^2} + \frac{1}{8(Z-1)} - \frac{1}{16} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.10)$$

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For $0 < |z-1| < 2$. Because the largest inverse (negative) power is three, the singularity at z =1 is called a third-order pole; the value of the residue
e is 1/8. Generally, I refer to a first-order pole as a simple pole. The construction of a *Laurent expansion* is not the method of choice in computing a residue. (For an essential singularity, it is the only method; however, essential singularities are very rare in applications.)
The common method for a pole of order n is

$$Res\,[\,f(z);\,z_j\,] = \frac{1}{(n-1)!}\lim_{z\to z_j}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_j)^n f(z)\right] \ldots\ldots\ldots\ldots\ldots\ldots (3.11)$$

For a simple pole equation (3.11) simplifies to

$$Res\,[\,f(z);\,z_j\,] = \lim_{z\to z_j}\left[(z-z_j)\;\; f(z)\right] \ldots\ldots\ldots\ldots\ldots\ldots (3.12)$$

Quite often, $f(z) = p(z)/q(z)$. from l'Hospital's rule, it follows that

$$Res\,[\,f(z);\,z_j\,] = \frac{p(z_j)}{q(z_j)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.13)$$

Laplace's And Poisson's Equations

Using Laplace transforms to solve Laplace's or Poisson's equations would appear to be a strange choice since there are no initial. However, for half and quarter plane problems, one (or both) of the independent variables can act as the variable. The tricky part is satisfying the boundary conditions. The following example shows how this is done.

Example 8

Let us solve Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = xe^{-x}, \quad 0 < x < \infty,\; 0 < y < a \ldots\ldots\ldots\ldots\ldots\ldots (4.1)$$

Subject to the boundary conditions

$$u(0,y) = 0, \qquad \lim_{x\to\infty} |u(x,y)\,| < \infty,\, 0 < y < a \;\ldots\ldots\ldots\ldots\ldots\ldots (4.2)$$

And

$$u(x,0) = 0,\;\; u_x(x,a) = 0 \qquad 0 < x < \infty \;\ldots\ldots\ldots\ldots\ldots\ldots (4.3)$$

This problem gives the electrostatic potential within a semi-infinite slab of thickness a with a charge density xe^{-x}.
Because the domain is semi-infinite the $x - direction$, I introduce the Laplace transform

$$U(s,y) = \int_0^\infty u(x,y)e^{-sx}dx \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4.4)$$

Thus, taking the Laplace transform of the equation (4.1), I have

$$\frac{d^2U(s,y)}{dy^2} + s^2U(s,y) - su(0,y) - u_x(0,y) = \frac{1}{(s+1)^2} \ldots\ldots\ldots\ldots\ldots\ldots (4.5)$$

Although $u(0,y) = 0$, $u_x(0,y)$ is unknown and I denote its value by *f(y)*, therefore the equation (4.5) becomes

$$\frac{d^2U(s,y)}{dy^2} + s^2U(s,y) = f(y) + \frac{1}{(s+1)^2}\;, \quad 0 < y < a \ldots\ldots\ldots\ldots\ldots\ldots (4.6)$$

With $U(s,0) = U'(s,a) = 0$.
To solve the equation, (4.6), I first assume that I can rewrite *f(y)* as the Fourier series
I now understand why I rewrote the right side of the equation(4.6) as a Fourier series; the solution *U(s,y)* automatically satisfies the boundary condition

$$U(s,0) = U'(s,a) = 0.$$

Example 9

Let us solve a similar problem to the previous one, but in cylindrical coordinates. Here,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = \frac{2}{b}\mathrm{n}(z)\;\delta\;(\mathrm{r}-\mathrm{b})\;0 \le r < a\,, \quad 0 < z < \infty \;\ldots (4.7)$$

Subject to the boundary conditions

$$u(r,0) = 0 \;\;, \lim_{z\to\infty}|u(r,z)| < \infty, \qquad 0 \le r < a \;\ldots\ldots\ldots\ldots (4.8)$$

and

$$u(a,z) = 0, \qquad 0 < z < \infty\,, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4.9)$$

Where $0 < b < a$ .This problem gives the electrostatic potential within a semi-infinite cylinder of radius a that is grounded and has the charge density of *n(z)* within an infinitesimally thin shell located at *r = b*.
Because the domain is semi-infinite in the z direction, I introduce the Laplace transform

$$U(r,s) = \int_0^\infty u(r,z)e^{-sz}dz \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.10)$$

Thus, taking the Laplace transform of the equation (4.7), I have

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$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dU(r,s)}{dr}\right] + s^2U(r,s) - su(r,0) - u_z(r,0) = \frac{2}{b}N(s)\,\delta\,(r-b)\ldots\ldots.(4.11)$$

Although $u(r,0) = 0$, $u_z(r,0)$ is unknown and I denote its value by *f(r);* therefore, the equation (4.11) becomes

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dU(r,s)}{dr}\right] + s^2U(r,s) = f(r) + \frac{2}{b}N(s)\,\delta\,(r-b)\ \ldots\ldots\ldots(4.12)$$

With $\lim_{r\to 0}|u(r,s)| < \infty, \quad and\ U(a,s) = 0.$

Conclusion

The Laplace transform is an efficient and elegant method for solving certain classes of partial differential equations, and is a robust technique for solving linear partial differential equations with well-defined initial and boundary conditions. Its ability to transform time derivatives into algebraic terms makes it a valuable tool for engineers and scientists dealing with heat conduction, wave propagation, and diffusion problems. By converting time derivatives into algebraic terms, it simplifies the solution process and offers clear analytical pathways. Its use remains widespread in engineering and applied mathematics, particularly for time-dependent problems.

Conflict of interest. Nil

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